Static spacetimes with prescribed multipole moments; a proof of a conjecture by Geroch

Magnus Herberthson

Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden.
e-mail: maher@mai.liu.se

Abstract. In this paper we give sufficient conditions on a sequence of multipole moments for a static spacetime to exist with precisely these moments. The proof is constructive in the sense that a metric having prescribed multipole moments up to a given order can be calculated. Since these sufficient conditions agree with already known necessary conditions, this completes the proof of a long standing conjecture due to Geroch.

1. Introduction

There are various definitions of multipole moments, [9]. For static asymptotically flat spacetimes, the standard definition was given by Geroch [2], and this definition was later extended to the stationary case by Hansen [4]. In essence, the recursive definition by Geroch from [2], is a non-trivial modification of the Newtonian definition taking place in a conformally rescaled spacetime, see Section 2. Since the conformal factor involved is not unique the relativistic definition must, apart from taking into account the curvature terms, ensure invariance under the conformal freedom available.

In [2], Geroch made two conjectures, namely (i^0 will be defined shortly)

Conjecture1: Two static solutions of Einstein's equations having identical multipole moments coincide, at least in some neighborhood of i^0.

and

Conjecture 2: Given any set of multipole moments, subject to the appropriate convergence condition, there exists a static spacetime of Einstein's equations having precisely those moments.

Conjecture 1 was proved in [1] and although many properties of the multipoles are known, see e.g. [9], Conjecture 2 has not yet been proven. There are partial results however. For instance, it is known [10] that an arbitrary sequence of multipole moments uniquely defines formal power series of relevant field variables, and that if the series converge, this give a static spacetime having these moments. In [3], on the other hand, growth conditions are given on a set referred to as *null data*, which ensures the existence of a static space time possessing these null data. The null data are shown to

be in a one-to-one correspondence with the Geroch multipole moments (with non-zero monopole), but since this correspondence is rather implicit, no growth condition on the actual multipole moments are derived. On the "necessary side", the situation is more satisfactory. In [8], it was shown that the multipole moments of any asymptotically flat stationary, and therefore static, spacetime do not grow "too fast", and precise conditions were given.

The purpose of this paper is to show that the necessary conditions in [8] are also sufficient, i.e., if they are satisfied, there exists a static asymptotically flat vacuum spacetime with these moments. Therefore, this will prove Geroch's Conjecture 2 above. The proof in Section 3 will include an explicit recursion of the desired metric, which means that the metric for a static spacetime with prescribed moments up to an arbitrary order can be calculated. We will first prove that the metric cast in a special form is uniquely defined by the moments, at least formally, and then show that the result from [3] can be used to deduce convergence of the series for the metric components.

One major obstacle has been to explicitly link a certain metric and potential with a given set of multipole moments. This problem goes back to the actual definition of the moments, i.e., the recursion (2), where the operation of "taking the totally symmetric and trace-free part" effectively obscures the relation between the metric, the potential and the moments, unless some care is taken. Another issue is the coordinate freedom. Without extra restrictions on the metric it is impossible to uniquely derive the metric components from the moments since one can always change coordinates and hence the components of the metric. However, as will be shown in the following sections, these two issues can be addressed simultaneously.

We also remark that the sufficient conditions given do not contain the usual requirement that the mass term (monopole) m is non-zero, although that is a condition which may be required for physical reasons.

2. Multipole moments of stationary spacetimes

Although in this paper, we address a conjecture concerning static spacetimes, we have chosen to formulate the conjecture within the stationary setting. One reason is that we will refer to results from [8], where this setting is used. Therefore, in this section we quote the definition of multipole moments given by Hansen in [4], which is an extension to stationary spacetimes of the definition for static spacetimes by Geroch [2]. We thus consider a stationary spacetime (M, g_{ab}) with signature $(-,+,+,+)$ and with a timelike Killing vector field ξ^a. We let $\lambda = -\xi^a \xi_a$ be the norm, and define the twist ω through $\nabla_a \omega = \epsilon_{abcd} \xi^b \nabla^c \xi^d$. If V is the 3-manifold of trajectories, the metric g_{ab} induces the positive definite metric

$$h_{ab} = \lambda g_{ab} + \xi_a \xi_b$$

on V. It is required that V is asymptotically flat, i.e., there exists a 3-manifold $\hat{V}$ and a conformal factor Ω satisfying

(i) $\hat{V} = V \cup i^0$, where i^0 is a single point

(ii) $\hat{h}_{ab} = \Omega^2 h_{ab}$ is a smooth metric on $\hat{V}$

(iii) At i^0, $\Omega = 0, \hat{D}_a\Omega = 0, \hat{D}_a\hat{D}_b\Omega = 2\hat{h}_{ab}$,

where $\hat{D}_a$ is the derivative operator associated with $\hat{h}_{ab}$. i^0 is referred to as spacelike infinity‡. On M, and/or V, one defines the scalar potential

$$\phi = \phi_M + i\phi_J, \quad \phi_M = \frac{\lambda^2 + \omega^2 - 1}{4\lambda}, \; \phi_J = \frac{\omega}{2\lambda}. \tag{1}$$

The multipole moments of M are then defined on $\hat{V}$ as certain derivatives of the scalar potential $\hat{\phi} = \phi/\sqrt{\Omega}$ at i^0. More explicitly, following [4], let $\hat{R}_{ab}$ denote the Ricci tensor of $\hat{V}$, and let $P = \hat{\phi}$. Define the sequence $P, P_{a_1}, P_{a_1a_2}, \ldots$ of tensors recursively:

$$P_{a_1\ldots a_n} = C[\hat{D}_{a_1}P_{a_2\ldots a_n} - \frac{(n-1)(2n-3)}{2}\hat{R}_{a_1a_2}P_{a_3\ldots a_n}], \tag{2}$$

where $C[\,\cdot\,]$ stands for taking the totally symmetric and trace-free part. The multipole moments of M are then defined as the tensors $P_{a_1\ldots a_n}$ at i^0.

In [2], a slightly different setup and a different potential is used, but it is known [10] that the potential used there and (1) with $\omega = 0$ produce the same moments.

3. Static spacetimes with prescribed multipole moments

In this section, we will formulate and prove the desired theorem. The theorem will be as conjectured in [8], i.e., in essence that if the sequence of multipole moments is naturally connected to a harmonic function on $\mathbf{R}^3$, there exists a static asymptotically flat vacuum spacetime having precisely those moments. Namely, (cf. Theorem 8 in [8]), consider $\mathbf{R}^3$ with Cartesian coordinates $\mathbf{r} = (x, y, z) = (x^1, x^2, x^3)$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. With the convention that, in terms of components, $P^0_\alpha = P^0_{\underbrace{11\cdots1}_{\alpha_1}\underbrace{22\cdots2}_{\alpha_2}\underbrace{33\cdots3}_{\alpha_3}}$, we have the following theorem.

Theorem 1. *Let $P^0, P^0_{a_1}, P^0_{a_1a_2}, \ldots$ be a sequence of real valued totally symmetric and trace free tensors on $\mathbf{R}^3$, and let $P^0, P^0_{i_1}, P^0_{i_1i_2}, \ldots$ be the corresponding components with respect to the Cartesian coordinates $\mathbf{r} = (x, y, z)$. If $u(\mathbf{r}) = \sum_{|\alpha|\geq 0} \frac{\mathbf{r}^\alpha}{\alpha!}P^0_\alpha$ converges in a neighbourhood of the origin in $\mathbf{R}^3$, there exists a static asymptotically flat vacuum spacetime having the moments $P^0, P^0_{a_1}, P^0_{a_1a_2}, \ldots$.*

Note that we do not require the monopole $P^0 = \hat{\phi}(0)$ to be non-zero§. The proof, however, will first be carried out under the assumptions $P^0 \neq 0$, and this condition will then be relaxed in Section 3.8. The proof for the case $P^0 \neq 0$ will be carried out in a sequence of lemmas (Lemma 2 - Lemma 13), which also show that the metric up to a given order can be calculated explicitly, but first we will give an outline of the proof, discuss the notation and formulate the relevant field equations.

‡ i^0 is also used in a four-dimensional context.

§ Due to the definition of ϕ, $P^0 = -m$, where m is the mass of the spacetime.

3.1. Outline of the proof

As mentioned in Section 1, it is possible to reduce the coordinate freedom in the metric components, and simultaneously establish a direct link between the potential $\hat{\phi}$ and the desired moments. This will be done in Section 3.4, where the link is expressed in Theorem 8.

In Section 3.5, we will address the conformal field equations from Section 3.3, and see that the form of the metric given in Section 3.4 results in singular equations. By requiring that the conformal field equations are smooth at i^0, further restriction will be put on the rescaled metric $\hat{h}_{ij}$, which then takes its final form.

With the form of the metric fixed, we will in Section 3.6 show that the field equations determine the metric components as a formal power series. More precisely, we will show that when the monopole is non-vanishing, a certain subset of the field equations is sufficient to specify the metric completely.

In Section 3.7 we will address the full set of equations, as well as the issue of convergence of the power series derived. By referring to a result by Friedrich, [3], convergence of the power series will be concluded. It will also be seen that the full set of equations are satisfied.

Finally, in Section 3.8, we will relax the condition that the monopole is nonzero.

3.2. Notation

Small Latin letters $a, b, \ldots$ refer to abstract indices, as in Section 2. Small Latin letters $i, j, k, \ldots$ are numerical indices and refer to components with respect to the normal coordinates $(x, y, z) = (x^1, x^2, x^3)$ introduced below. Since these components refer to this particular coordinate system only, the equations will not be tensor equations. With this said, we will still use $=$ instead of $\doteq$.

Almost all variables‖ are assumed to be formally analytic, i.e., they admit a formal power series expansion (again in terms of the chosen coordinates), so that, for a tensor, A_{ijk} say, we can write

$$A_{ijk} = \sum_{n=0}^{\infty} A_{ijk}^{[n]}$$

where $A_{ijk}^{[n]}$ denots polynomials in (x, y, z) which are homogeneous of order n (and where the summation may be formal). Both η_{ij} and η^{ij} denotes the identity matrix, and by $[A_{ij}]$ we denote the trace of A_{ij}, i.e., $\eta^{ij} A_{ij}$. We also use $\partial_i = \frac{\partial}{\partial x^i}$.

With $r = \sqrt{x^2 + y^2 + z^2}$, $f_1 \equiv f_2 \pmod{r^2}$ means that $f_1(x, y, z) - f_2(x, y, z) = r^2\ g(x, y, z)$ for some (formally analytic) function g. When $f \equiv 0 \pmod{r^2}$, so that $f = r^2\ g$ for some formally analytic function g, we also use the shorter notation $r^2 | f$.

In the proof of Lemma 13 we will refer to [3] and hence use some of the notation there.

‖ The exception is $r = \sqrt{x^2 + y^2 + z^2}$

3.3. The field equations

Apart from having the correct multipole moments, we must ensure that the metric describes a static vacuum spacetime. We will formulate our equations on the 3-manifold $(\hat{V}, \hat{h}_{ab})$ defined in Section 2, starting with the field equations from [2]. However, the 3-manifolds in [2] and [4] are defined slightly differently, and the relations imply the following.

Starting with a static spacetime (M, g_{ab}) with timelike killingvector ξ^a, we put $0 < \lambda = -\xi^a\xi_a$ and $\Psi = 1 - \sqrt{\lambda}$. From [2] we consider¶ a 3-surface V_G orthogonal to ξ^a. In terms of the metric on V_G: $(h_G)_{ab} = g_{ab} + \xi_a\xi_b/\lambda$, the field equations are, [2],

$$\left\{\begin{array}{c} (D_G)^a(D_G)_a\Psi = 0 \\ (R_G)_{ab} = \frac{1}{\Psi-1}(D_G)_a(D_G)_b\Psi \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} (R_G)^a{}_a = 0 \\ (R_G)_{ab} = \frac{1}{\sqrt{\lambda}}(D_G)_a(D_G)_b\sqrt{\lambda} \end{array}\right. , \tag{3}$$

where $(D_G)_a$ is the derivative operator and $(R_G)_{ab}$ is the Ricci tensor associated with $(V_G, (h_G)_{ab})$.

On the 3-manifold V on the other hand, the metric is $h_{ab} = \lambda g_{ab} + \xi_a\xi_b$, i.e., $h_{ab} = \lambda(h_G)_{ab}$, and with $\hat{h}_{ab} = \Omega^2 h_{ab}$, this implies that $\hat{h}_{ab} = (\sqrt{\lambda}\Omega)^2(h_G)_{ab}$. We can now express equations (3) on $(\hat{V}, \hat{h}_{ab})$ using as conformal factor $\hat{\Omega} = \sqrt{\lambda}\Omega$. Using the properties of conformal transformations, [12], we find that (3) becomes

$$\begin{array}{lcl} \hat{R} + 4\hat{h}^{ab}\hat{D}_a\hat{D}_b\ln(\sqrt{\lambda}\Omega) - 2\hat{h}^{ab}\hat{D}_a\ln(\sqrt{\lambda}\Omega)\hat{D}_b\ln(\sqrt{\lambda}\Omega) & = & 0 \\ \hat{R}_{ab} + \hat{D}_a\hat{D}_b\ln(\sqrt{\lambda}\Omega) + \hat{h}_{ab}\hat{h}^{cd}\hat{D}_c\hat{D}_c\ln(\sqrt{\lambda}\Omega) + & & \\ \hat{D}_a\ln(\sqrt{\lambda}\Omega)\hat{D}_b\ln(\sqrt{\lambda}\Omega) - \hat{h}_{ab}\hat{h}^{cd}\hat{D}_c\ln(\sqrt{\lambda}\Omega)\hat{D}_c\ln(\sqrt{\lambda}\Omega) & = & \\ \frac{1}{\sqrt{\lambda}}[\hat{D}_a\hat{D}_b\sqrt{\lambda} + \hat{D}_a\ln(\sqrt{\lambda}\Omega)\hat{D}_b\sqrt{\lambda} + & & \\ \hat{D}_b\ln(\sqrt{\lambda}\Omega)\hat{D}_a\sqrt{\lambda} - \hat{h}_{ab}\hat{h}^{cd}\hat{D}_d\ln(\sqrt{\lambda}\Omega)\hat{D}_c\sqrt{\lambda}]. & & \end{array} \tag{4}$$

Thus, we are looking for a metric $\hat{h}_{ab}$ on $\hat{V}$, defined in a neighbourhood of i^0, which satisfies (4) and where the corresponding 4-dimensional spacetime has prescribed multipole moments. As mentioned earlier, the equation (4) cannot have a unique solution in terms of coordinates. This is not only due to the fact that one can always represent a metric in different coordinate systems, but also because there is a freedom in the conformal factor Ω. In the next section we will put the metric $\hat{h}_{ab}$ in a canonical form, i.e., use a preferred coordinate system. This coordinate system is also constructed so that it allows specification of the multipole moments, i.e., puts (2) in a form which is more transparent.

3.4. Prescribed moments and canonical form of the metric

In this section we will, in a sense, reverse the arguments from [8]. The key point is to work in normal coordinates, in which the Ricci tensor $\hat{R}_{ab}$ takes a special form. This will lead to Theorem 8, which allows for a direct connection between the potential $P = \hat{\phi}$ and the multipole moments.

¶ In [2], the potential $\psi = -\Psi$ is used.

Let us therefore first introduce normal coordinates $(x, y, z) = (x^1, x^2, x^3)$ on $\hat{V}$, where the point i^0 has coordinates $\mathbf{0}$, and such that (x, y, z) "are Cartesian" at i^0, i.e., in terms of the coordinates, $\hat{h}_{ij}(i^0) = \eta_{ij}$. By assumption, $\hat{h}_{ab}$, (and therefore $\hat{R}_{ab}$, $\hat{R}$) and also the conformal factor Ω are at this stage formal power series, while u from Theorem 1 is given by a power series which converges in a neighbourhood of $\mathbf{0}$. u is defined on $\mathbf{R}^3$, but in terms of the normal coordinates introduced, the sum $\sum_{|\alpha|\geq 0} \frac{\mathbf{r}^\alpha}{\alpha!} P^0_\alpha$ can also be interpreted on $\hat{V}$, namely as the potential function $\hat{\phi}(\mathbf{r})$.

We now turn to the condition on $\hat{R}_{ab}$. The condition is in [8] formulated through the complexification $\hat{V}_{\mathbf{C}}$ of $\hat{V}$, and by using complex null vectors η^a, i.e., vectors η^a with $\eta^a\eta_a = 0$. Using normal coordinates, it was shown in [8] that for any fixed φ, the complex curve

$$\gamma_\varphi : t \to (t\cos\varphi, t\sin\varphi, it),$$

has tangent vector $\eta^a = \eta^a_\varphi(t) = \cos\varphi(\frac{\partial}{\partial x^1})^a + \sin\varphi(\frac{\partial}{\partial x^2})^a + i(\frac{\partial}{\partial x^3})^a$ which satisfies $\eta^a\eta_a = 0$. The relevant condition on $\hat{R}_{ab}$ is then to be found in Lemma 6 of [8], where the condition $\tilde{\eta}^a\tilde{\eta}^b\tilde{R}_{ab} = 0$ was made. With our notation this reads $\eta^a\eta^b\hat{R}_{ab} = 0$, and it is this condition, together with the normality of the coordinates, which allows for a direct connection between the potential $\hat{\phi}$ and the moments (Theorems 7 and 8 in [8], Theorem 8 below). Although the results in [8], which build on the techniques developed in [5, 6, 7], depend crucially on complex quantities (especially the concept of "leading term") it is interesting to note that all of these tools and arguments can be given in purely real terms. Thus, rather then referring to, and reversing, the arguments in [8], we will derive/translate the corresponding conclusion using only real quantities.

The first property is the analogue of Lemma 1c in [8], namely, with η^a as above, that $\eta^{a_1}\ldots\eta^{a_n}T_{a_1\ldots a_n} = \eta^{a_1}\ldots\eta^{a_n}C[T_{a_1\ldots a_n}]$, although we here state it in a vector space using the radius vector.

Lemma 2. *Let (x, y, z) be Cartesian coordinates on $\mathbf{R}^3$, let $\mathbf{r}^a = x(\frac{\partial}{\partial x})^a + y(\frac{\partial}{\partial y})^a + z(\frac{\partial}{\partial z})^a$, and put $r = \sqrt{x^2+y^2+z^2}$. Then, for any tensor $T_{a_1\ldots a_n}$, $\mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}C[T_{a_1\ldots a_n}] \equiv \mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}T_{a_1\ldots a_n} \pmod{r^2}$*

Proof. $C[T_{a_1\ldots a_n}]$ is constructed from $T_{a_1\ldots a_n}$ by taking the totally symmetric part and subtracting suitable amounts of the tensors $\eta_{(a_1a_2}T_{a_3\ldots a_n)}{}^b{}_b$, $\eta_{(a_1a_2}\eta_{a_3a_4}T_{a_5\ldots a_n)}{}^{b}{}_{b}{}^{c}{}_{c}$, $\ldots$ where η_{ab} is the Euclidean metric on $\mathbf{R}^3$. Thus, it is clear that $C[T_{a_1\ldots a_n}] = T_{(a_1\ldots a_n)} + \eta_{(a_1a_2}S_{a_3a_4\ldots a_n)}$ for some tensor $S_{a_3a_4\ldots a_n}$. Since $\mathbf{r}^a\mathbf{r}^b\eta_{ab} = r^2$, the statement follows by transvecting with $\mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}$ and observing that $\mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}T_{(a_1\ldots a_n)} = \mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}T_{a_1\ldots a_n}$. □

This property can also be realized in the following way. It is clear that $\mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}T_{a_1\ldots a_n}$ is a homogeneous polynomial $p = p(x, y, z)$ of degree n. Using spherical coordinates in $\mathbf{R}^3$, i.e., $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we express p in spherical harmonics Y_l^m and get $p = \sum_{l=0,l \text{ even}}^{n} \sum_{m=-l}^{l} r^n c_l^m Y_l^m(\theta, \phi)$ if n is even.

If $l = n-2$ or less, $\sum_{m=-l}^{l} r^n c_l^m Y_l^m(\theta,\phi) = r^{n-l} \sum_{m=-l}^{l} r^l c_l^m Y_l^m(\theta,\phi)$ with r^{n-l} smooth and divisible by r^2, and where the sum is a smooth polynomial. This means that $\mathbf{r}^{a_1}\ldots\mathbf{r}^{a_n}C[T_{a_1\ldots a_n}]$ corresponds to the polynomial $p_C = \sum_{m=-n}^{n} r^n c_n^m Y_n^m(\theta,\phi)$, i.e., only the terms corresponding to $l = n$ are kept. Identical remarks hold if n is odd.

Using normal coordinates, the property of Lemma 2 can be carried over to the manifold $\hat{V}$. Namely, from [11] we have the following lemma, which we formulate for the particular case when $\hat{h}_{ij}(i^0) = \eta_{ij}$.

Lemma 3. *$(x,y,z) = (x^1,x^2,x^3)$ are normal coordinates on $(\hat{V},\hat{h}_{ab})$, if and only if $x^i\hat{h}_{ij}(x^k) = x^i\hat{h}_{ij}(i^0)$, i.e., $x^i\hat{h}_{ij} = x^i\eta_{ij}$.*

Thus, in normal coordinates, the above lemma implies that $x^ix^j\hat{h}_{ij} = x^2+y^2+z^2 = r^2$. This means that by the same arguments as above,

$$x^{i_1}\ldots x^{i_n}C[T_{i_1\ldots i_n}] \equiv x^{i_1}\ldots x^{i_n}T_{i_1\ldots i_n} \pmod{r^2} \text{ in } \hat{V} \tag{5}$$

Remark. That (5) is still true in $\hat{V}$ depends on the fact that we have $x^ix^j\hat{h}_{ij} = r^2$. However, in Lemma 2, we had a fixed tensor $T_{a_1\ldots a_n}$ on a vector space, while on $\hat{V}$, $T_{a_1\ldots a_n}$ is a tensor field. This does not affect the equality (5), but when evaluating at i^0, where $(x^1,x^2,x^3) = (0,0,0)$, it just says $0=0$. Another effect is that $x^{i_1}\ldots x^{i_n}T_{i_1\ldots i_n}$ is a polynomial which is the sum of a homogeneous polynomial of degree n and a polynomial containing only higher order terms. However, by replacing x^i with $\rho^i = x^i/r$, so that $\rho^i\rho^j\hat{h}_{ij} = 1$, both $\rho^{i_1}\ldots\rho^{i_n}C[T_{i_1\ldots i_n}]$ and $\rho^{i_1}\ldots\rho^{i_n}T_{i_1\ldots i_n}$ will be direction dependent quantities at i^0. In the limit $r \to 0$, the higher order terms vanish and we get

$$\begin{aligned} \rho^{i_1}\ldots\rho^{i_n}T_{i_1\ldots i_n}(i^0) &= \sum_{l=0,l \text{ even}}^{n} \sum_{m=-l}^{l} c_l^m Y_l^m(\theta,\phi) \Rightarrow \\ \rho^{i_1}\ldots\rho^{i_n}C[T_{i_1\ldots i_n}(i^0)] &= \sum_{m=-n}^{n} c_n^m Y_n^m(\theta,\phi), \end{aligned}$$

if n is even; and a corresponding relation when n is odd ($l = 1,3,5,\ldots n$). This last equality also tells us that $C[T_{i_1\ldots i_n}]$ transvected with $x^{i_1}\ldots x^{i_n}$ or $\rho^{i_1}\ldots\rho^{i_n}$ still contains the full information, since there are $2n+1$ degrees of freedom in the RHS.

In view of Lemma 2 and the corresponding property (5) on $\hat{V}$, it is obvious that $\mathbf{r}^a\mathbf{r}^b\hat{R}_{ab} \equiv 0 \pmod{r^2}$, i.e., that $r^2|\mathbf{r}^a\mathbf{r}^b\hat{R}_{ab}$ is a desirable property in order to simplify (2) since $\mathbf{r}^a\mathbf{r}^b\hat{R}_{ab}$ will then not affect the multipole moments. As it turns out, this condition can be formulated purely in algebraic terms. This will be done in Lemma 7 below. To prepare for this lemma, we need some more tools.

Lemma 4. *Suppose $A = A_{ij} = A_{(ij)}$ are the components of a symmetric tensor field A_{ab} on $\hat{V}$ with respect to normal coordinates $(x,y,z) = (x^1,x^2,x^3)$ and that A_{ij} has the property that $r^2|\eta^{ij}A_{ij}$, $A_{ij}x^i = 0$. Then A is uniquely decomposable as*

$$\begin{aligned} A &= f_1(x,y)A_1 + f_2(y,z)A_2 + f_3(x,z)A_3 + f_4(x,z)A_4 \\ &\quad + f_5(x,y,z)A_5 + f_6(x,y,z)A_6 + f_7(x,y,z)A_7, \text{ where} \end{aligned}$$

$$
\begin{aligned}
A_1 &= \begin{pmatrix} -y^2-z^2 & xy & xz \\ xy & -x^2-z^2 & yz \\ xz & yz & -x^2-y^2 \end{pmatrix}\\
A_2 &= \begin{pmatrix} 2z\left(y^2+z^2\right) & -xyz & -x\left(y^2+2z^2\right) \\ -xyz & 0 & x^2y \\ -x\left(y^2+2z^2\right) & x^2y & 2x^2z \end{pmatrix}\\
A_3 &= \begin{pmatrix} 2xz^2 & yz^2 & -\left(2x^2+y^2\right)z \\ yz^2 & 0 & -xyz \\ -\left(2x^2+y^2\right)z & -xyz & 2x\left(x^2+y^2\right) \end{pmatrix}\\
A_4 &= \begin{pmatrix} 0 & -xyz & xy^2 \\ -xyz & 2z\left(x^2+z^2\right) & -y\left(x^2+2z^2\right) \\ xy^2 & -y\left(x^2+2z^2\right) & 2y^2z \end{pmatrix}\\
A_5 &= \begin{pmatrix} 0 & -z\left(y^2+z^2\right) & y\left(y^2+z^2\right) \\ -z\left(y^2+z^2\right) & 2xyz & -x(y-z)(y+z) \\ y\left(y^2+z^2\right) & -x(y-z)(y+z) & -2xyz \end{pmatrix}\\
A_6 &= \begin{pmatrix} 2xyz & -z\left(x^2+z^2\right) & -y(x-z)(x+z) \\ -z\left(x^2+z^2\right) & 0 & x\left(x^2+z^2\right) \\ -y(x-z)(x+z) & x\left(x^2+z^2\right) & -2xyz \end{pmatrix}\\
A_7 &= \begin{pmatrix} 0 & xz^2 & -xyz \\ xz^2 & 2yz^2 & -\left(x^2+2y^2\right)z \\ -xyz & -\left(x^2+2y^2\right)z & 2y\left(x^2+y^2\right) \end{pmatrix}.
\end{aligned}
$$

Proof. In terms of matrices, the condition $r^2|\eta^{ij}A_{ij}$ is just that r^2 divides the trace of S, i.e., $r^2|[A]$, or $[A]\equiv 0 \pmod{r^2}$. Therefore, we can make the following ansatz

$$
A = \begin{pmatrix} X(x,y,z) & a(x,y,z) & b(x,y,z) \\ a(x,y,z) & Y(x,y,z) & c(x,y,z) \\ b(x,y,z) & c(x,y,z) & r^2Z(x,y,z)-X(x,y,z)-Y(x,y,z) \end{pmatrix}
$$

where a, b, c, X, Y and Y are analytic in the variables indicated. The condition $A_{ij}x^i = 0$ then translates to

$$
\begin{aligned}
xX(x,y,z)+ya(x,y,z)+zb(x,y,z) &= 0\\
xa(x,y,z)+yY(x,y,z)+zc(x,y,z) &= 0\\
xb(x,y,z)+yc(x,y,z)+z[r^2Z(x,y,z)-X(x,y,z)-Y(x,y,z)] &= 0
\end{aligned} \tag{6}
$$

We use analyticity of the functions involved, and start by writing

$$
\begin{aligned}
X(x,y,z) &= X_1(x)+y\,X_2(x)+y^2X_3(x,y)+z\,X_4(x,y,z),\\
Y(x,y,z) &= Y_1(y)+x\,Y_2(y)+x^2Y_3(x,y)+z\,Y_4(x,y,z),\\
a(x,y,z) &= a_1+x\,a_2(x)+y\,a_3(y)+xy\,f_1(x,y)+z\,a_4(x,y,z).
\end{aligned}
$$

Inserted in (6), the limits $z\to 0, y\to 0$ gives $X_1(x)=0, a_1=0, a_2(x)=0$, while the limits $z\to 0, x\to 0$ gives $Y_1(y)=0, a_3(y)=0$. Given this, $z\to 0$ implies $X_2(x)=0, Y_2(y)=0$, followed by $X_3(x,y)=-f_1(x,y)$ and $Y_3(x,y)=-f_1(x,y)$. A

further evaluation of (6) then shows that $b(x,y,z) = -x\,X_4(x,y,z) - y\,a_4(x,y,z)$ and $c(x,y,z) = -x\,a_4(x,y,z) - y\,Y_4(x,y,z)$. By writing

$$Y_4(x,y,z) = Y_5(x,z) + y\,Y_6(x,y,z) - z\,f_1(x,y),$$

(6) together with $y = 0$ shows that $Y_5(x,z)$ contains the factor x^2+z^2; and similarly

$$X_4(x,y,z) = X_5(y,z) + x\,X_6(x,y,z) - z\,f_1(x,y)$$

in (6) with $x = 0$ reveals that $X_5(y,z)$ contains y^2+z^2. Thus $Y_5(x,z) = 2(x^2+z^2)f_4(x,z)$ and $X_5(y,z) = 2(y^2+z^2)f_2(y,z)$. To proceed, we write

$$X_6(x,y,z) = X_8(x,z) + 2y\,f_6(x,y,z).$$

In addition, (6) with $y = 0$ then shows that $X_8(x,z) = 2z\,f_3(x,z)$ for some function f_3. Next, (6) implies $Z(x,y,z) = 2y\,f_7(x,y,z) - 2f_1(x,y) + 2x\,f_3(x,z) + 2z\,f_2(y,z) + 2z\,f_4(x,z)$, for some function f_7. Also, (6) with $x = 0$ then gives $Y_6(x,y,z) = 2z\,f_7(x,y,z) + 2x\,f_5(x,y,z)$, and a final application of (6) shows that $a_4(x,y,z) = y\ z\ f_3(x,z) - x\ y\ f_2(y,z) - x^2\ f_6(x,y,z) - z^2\ f_6(x,y,z) - x\ y\ f_4(x,z) - y^2\ f_5(x,y,z) - z^2\ f_5(x,y,z) + x\ z\ f_7(x,y,z)$. By collecting terms, we get Theorem 4. □

In the proof of Lemma 7 below, A_{ij} will be the difference $\hat{h}_{ij} - \eta_{ij}$, and since $\hat{R}_{ij}$ involves the Christoffel symbols, we need a corresponding property for the inverse metric $\hat{h}^{ij}$.

Lemma 5. *Suppose that, in normal coordinates, $\hat{h}_{ij}$ has the property that $A_{ij} = \hat{h}_{ij} - \eta_{ij}$ satisfies $r^2|\eta^{ij}A_{ij}$, $A_{ij}x^i = 0$. Then the same holds for the inverse $\hat{h}^{ij}$, i.e., with $B^{ij} = \hat{h}^{ij} - \eta^{ij}$, we have $r^2|\eta_{ij}B^{ij}$ and $B^{ik}\hat{h}_{jk}x^j = B^{ik}\eta_{jk}x^j = 0$.*

Proof. This is most easily seen in terms of matrices. Namely, it is easy to check that the matrices $A_1, A_2, \ldots, A_7$ from Lemma 4 have the property that also the products A_iA_j, $1 \le i,j \le 7$ satisfies the assumptions of Lemma 4. But this means that the the inverse $(I+A)^{-1} = I + B = I + \sum_{n=1}^{\infty}(-A)^n$ will be a sum of the identity operator I and terms which all have the properties of Lemma 4. □

Let us now define the operator

$$D = x^j\frac{\partial}{\partial x^j}.$$

The operator D has the important property that if $f(x,y,z)$ is a homogeneous polynomial of order n, $D(f) = nD(f)$. This is true, whether f is a scalar or tensor valued. Because of this property, it easily follows that

Lemma 6. *If A_{ij} has the properties of Lemma 4, the same properties hold for $D(A_{ij})$.*

Before we state and prove Lemma 7, we note that the definition of the Christoffel symbols $\Gamma^k{}_{ij}$ implies that in normal coordinates, $2x^j\Gamma^k{}_{ij} = \hat{h}^{km}D(\hat{h}_{im})$, and in particular $2x^j\Gamma^k{}_{kj} = \hat{h}^{jk}D(\hat{h}_{jk}) = D(\ln|\hat{h}|)$, where $|\hat{h}|$ denotes the determinant of $\hat{h}_{ij}$.

Lemma 7. *Suppose that* (x^1, x^2, x^3) *are normal coordinates on* $(\hat{V}, \hat{h}_{ab})$. *Then* $r^2|\eta^{ij}(\hat{h}_{ij} - \eta_{ij}) \Rightarrow r^2|\mathbf{r}^a\mathbf{r}^b\hat{R}_{ab}$.

Proof. Since (x^1, x^2, x^3) are normal coordinates, $\Gamma^i{}_{kl}x^kx^l = 0$. Using this (and the fact that $\hat{h}_{ij}x^i = \eta_{ij}x^i$) the definition of $\hat{R}_{ij}$ gives

$$x^ix^jR_{ij} = -x^i\frac{\partial}{\partial x^i}(x^j\Gamma^k{}_{kj}) - x^j\Gamma^k{}_{kj} - x^ix^j\Gamma^m{}_{kj}\Gamma^k{}_{mi}.$$

Some further manipulation leads to

$$4x^ix^jR_{ij} = -2\hat{h}^{km}D(\hat{h}_{km}) - D(\hat{h}^{km})D(\hat{h}_{km}) - 2\hat{h}^{km}D^2(\hat{h}_{km}) \tag{7}$$

With the notation that B stands for any matrix satisfying the properties of Lemma 4, which means that $D^2(B) = D(B) = B$, $I \cdot B = B$, $B \cdot B = B$, $B + B = B$, (7) reads

$$4x^ix^jR_{ij} = [-2(I+B)B - B \cdot B - 2(I+B)B] = [B].$$

Since $[B]$ is divisible by r^2, so is $4x^ix^jR_{ij}$. □

We will now return to the original recursion (2). By combining the previous lemmas, we have the following theorem, which allows for the direct connection between the moments $P_{a_1\ldots a_n}$ and the potential $\hat{\phi}$.

Theorem 8. *Let* (x^1, x^2, x^3) *be normal coordinates on* $\hat{V}$, $\hat{h}_{ij} - \eta_{ij}$ *satisfy the properties of Lemma* 4, *and let* $P_{a_1\ldots a_n}$ *be defined by the recursion (*2*). Then*

$$x^{i_1}\ldots x^{i_n}P_{i_1\ldots i_n} \equiv x^{i_1}\ldots x^{i_n}\partial_{i_1}\cdots\partial_{i_n}P \pmod{r^2}$$

Proof. Put $c_n = \frac{n(2n-1)}{2}$. Then

$$\begin{aligned} x^{i_1}\ldots x^{i_n}P_{i_1\ldots i_n} &= x^{i_1}\ldots x^{i_n}C[\hat{D}_{i_1}P_{i_2\ldots i_n} - c_{n-1}\hat{R}_{i_1i_2}P_{i_3\ldots i_n}] \\ &= x^{i_1}\ldots x^{i_n}C[\hat{D}_{i_1}P_{i_2\ldots i_n}] - c_{n-1}x^{i_1}\ldots x^{i_n}C[\hat{R}_{i_1i_2}P_{i_3\ldots i_n}] \\ &\equiv x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}P_{i_2\ldots i_n} - c_{n-1}x^{i_1}\ldots x^{i_n}\hat{R}_{i_1i_2}P_{i_3\ldots i_n} \\ &\equiv x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}P_{i_2\ldots i_n} \pmod{r^2} \end{aligned}$$

Now,

$$x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}P_{i_2\ldots i_n} = x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}C[\hat{D}_{i_2}P_{i_3\ldots i_n} - c_{n-2}\hat{R}_{i_2i_3}P_{i_4\ldots i_n}]$$

and for some tensor $S_{i_4\ldots i_n}$

$$\begin{aligned} x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}C[\hat{D}_{i_2}P_{i_3\ldots i_n}] &= x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}(\hat{D}_{(i_2}P_{i_3\ldots i_n)} - \hat{h}_{(i_2i_3}S_{i_4\ldots i_n)}) \\ &= x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}\hat{D}_{i_2}P_{i_3\ldots i_n} - x^{i_1}\ldots x^{i_n}\hat{h}_{i_2i_3}\hat{D}_{i_1}S_{i_4\ldots i_n} \\ &\equiv x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}\hat{D}_{i_2}P_{i_3\ldots i_n} \pmod{r^2}, \end{aligned}$$

while

$$\begin{aligned} x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}C[\hat{R}_{i_2i_3}P_{i_4\ldots i_n}] &\equiv x^{i_1}\ldots x^{i_n}\hat{D}_{i_1}(\hat{R}_{i_2i_3}P_{i_4\ldots i_n}) \\ &\equiv x^{i_1}\ldots x^{i_n}(\hat{D}_{i_1}\hat{R}_{i_2i_3})P_{i_4\ldots i_n} \equiv 0 \pmod{r^2} \end{aligned}$$

where, in the last step, we have used that for some function f, $x^i x^j x^k \hat{D}_i \hat{R}_{jk} = x^i \hat{D}_i(x^j x^k \hat{R}_{jk}) - \hat{R}_{jk} x^i \hat{D}_i(x^j x^k) = D(r^2 f) - \hat{R}_{jk} x^i \partial_i (x^j x^k) + R_{jk} x^i \Gamma^j{}_{im} x^m x^k + R_{jk} x^i \Gamma^k{}_{im} x^j x^m = 2r^2 D(f) + r^2 D(f) - 2\hat{R}_{jk} x^j x^k \equiv 0 \pmod{r^2}$. Thus,

$$x^{i_1} \dots x^{i_n} P_{i_1 \dots i_n} \equiv x^{i_1} \dots x^{i_n} \hat{D}_{i_1} \hat{D}_{i_2} P_{i_3 \dots i_n} \pmod{r^2}.$$

Proceeding in the same way, we find that

$$x^{i_1} \dots x^{i_n} P_{i_1 \dots i_n} \equiv x^{i_1} \dots x^{i_n} \hat{D}_{i_1} \cdots \hat{D}_{i_n} P \pmod{r^2}.$$

Moreover,

$$\begin{aligned}
& x^{i_1} \dots x^{i_n} \hat{D}_{i_1} \dots \hat{D}_{i_n} P \\
&= x^{i_1} \dots x^{i_n} \partial_{i_1} \hat{D}_{i_2} \dots \hat{D}_{i_n} P - x^{i_1} \dots x^{i_n} \textstyle\sum_{m=2}^{n} \Gamma^m{}_{i_1 i_m} \hat{D}_{i_2} \dots \hat{D}_m \cdots \hat{D}_{i_n} P \\
&= x^{i_1} \dots x^{i_n} \partial_{i_1} \hat{D}_{i_2} \dots \hat{D}_{i_n} P \\
&= x^{i_1} \dots x^{i_n} \partial_{i_1} \partial_{i_2} \hat{D}_{i_3} \dots \hat{D}_{i_n} P - x^{i_1} \dots x^{i_n} \partial_{i_1} \textstyle\sum_{m=3}^{n} \Gamma^m{}_{i_1 i_m} \hat{D}_{i_3} \dots \hat{D}_m \dots \hat{D}_{i_n} P \\
&= x^{i_1} \dots x^{i_n} \partial_{i_1} \partial_{i_2} \hat{D}_{i_3} \cdots \hat{D}_{i_n} P
\end{aligned}$$

where, in the last step, we have used $x^i x^j x^k \partial_i \Gamma^m{}_{jk} = x^i \partial_i (x^j x^k \Gamma^m{}_{jk})$ $-x^i \partial_i (x^j x^k) \Gamma^m{}_{jk} = 0$. Again we can proceed and get the statement of the theorem. □

Remark. This theorem is comparable to Theorem 7 in [8]. The difference lies in the presentation since [8] uses complex vectors, and the effect is that the equivalence $\pmod{r^2}$ here becomes equality in [8] due to the fact that $r^2 = 0$ along the null vectors there. Also, although [8] uses null vectors, they are 'complexified unit vectors', while the statement here uses vectors x^i which are not normalized. However, in the statement of Theorem 8, one can (as commented before) replace each x^k by the direction dependent unit vector $\rho^k = x^k/r$.

Comparing with the definition of $P = \hat{\phi} = \sum_{|\alpha| \geq 0} \frac{\mathbf{r}^\alpha}{\alpha!} P^0_\alpha$ in the beginning of the section, Taylor's theorem together with Theorem 8 and the remark after Lemma 3 tells us that the multipoles produced will be precisely the desired multipoles P^0_α, and we may note that by the arguments presented, the multipoles will be unaffected by a change $\hat{\phi} \to \hat{\phi} + r^2 \gamma$.

Thus, by specifying $\hat{\phi}$, and by requiring that the metric $\hat{h}_{ij}$ be as in Theorem 8, the recursion (2) produces the prescribed multipoles moments. The issue is now whether (4) produces a power series for such a $\hat{h}_{ij}$, and furthermore if this series converges.

3.5. The conformal field equations

In this section, we will address the conformal field equations (4). The requirement that these equations extend smoothly to i^0 will put further restrictions on the form of the metric, and also involve the conformal factor Ω.

We consider $\hat{\phi}$ as fixed and real analytic with respect to the normal coordinates (x, y, z) in a neighbourhood of $i^0 = (0, 0, 0)$. From Section 2, we have that $(\omega = 0)$

$$\phi = \frac{\lambda^2 - 1}{4\lambda}, \quad P = \hat{\phi} = \phi/\sqrt{\Omega},$$

where λ, which is the norm of the Killing vector, and the conformal factor Ω appear in (4). The conformal factor Ω which must satisfy the conditions in Section 2, is not unique. Rather, we have the freedom $\Omega \to \Omega e^{\kappa}$, where κ is a formal power series which vanish at i^0. Also, $\hat{D}_a\kappa(0)$ is known to mix the moments, corresponding to a 'translation' in the classical sense. It is therefore natural to demand $\hat{D}_a\kappa(0) = 0$. Solving for λ, we thus get

$$\lambda = \sqrt{1 + 4\Omega\hat{\phi}^2} + 2\sqrt{\Omega}\hat{\phi}, \quad \Omega = r^2 e^{2\kappa}, \kappa(0) = \hat{D}_a\kappa(0) = 0.$$

It is important to note that λ is not even formally smooth, i.e., we cannot regard λ as a formal power series. This is of course due to the occurrence of $\sqrt{\Omega} = re^{\kappa}$, where $r = \sqrt{x^2 + y^2 + z^2}$ is non-regular at i^0. In effect, this will mean that each of the equations in (4) will split into two, seemingly doubling the number of equations.

To address (4) we split $\ln(\sqrt{\lambda}\Omega) = \frac{1}{2}\ln\lambda + \ln(r^2 e^{2\kappa})$ and note the convenient relation

$$\frac{1}{2}\hat{D}_a \ln\lambda = \frac{\hat{D}_a(r\hat{\phi}e^{\kappa})}{\sqrt{1 + 4r^2\hat{\phi}^2 e^{2\kappa}}}$$

Inserted in (4), this gives the equations

$$\begin{aligned}
&\hat{R}_{ab} + \hat{D}_a\hat{D}_b \ln(r^2 e^{2\kappa}) + \hat{D}_a \ln(r^2 e^{2\kappa})\hat{D}_b \ln(r^2 e^{2\kappa}) \\
&+\hat{h}_{ab}\hat{h}^{de}\hat{D}_d\hat{D}_e \ln(r^2 e^{2\kappa}) - \hat{h}_{ab}\hat{h}^{de}\hat{D}_d \ln(r^2 e^{2\kappa})\hat{D}_e \ln(r^2 e^{2\kappa}) \\
&+\hat{h}_{ab}\hat{h}^{de}\hat{D}_d \tfrac{\hat{D}_e(r\hat{\phi}e^{\kappa})}{\sqrt{1+4r^2\hat{\phi}^2 e^{2\kappa}}} - \hat{h}_{ab}\hat{h}^{de} \tfrac{\hat{D}_d(r\hat{\phi}e^{\kappa})\hat{D}_e \ln(r^2 e^{2\kappa})}{\sqrt{1+4r^2\hat{\phi}^2 e^{2\kappa}}} \\
&-2\tfrac{\hat{D}_a(r\hat{\phi}e^{\kappa})\hat{D}_b(r\hat{\phi}e^{\kappa})}{1+4r^2\hat{\phi}^2 e^{2\kappa}} = 0.
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\hat{R} + 4\hat{h}^{ab}\hat{D}_a\hat{D}_b \ln(r^2 e^{2\kappa}) + 4\hat{h}^{ab}\hat{D}_a \tfrac{\hat{D}_b(r\hat{\phi}e^{\kappa})}{\sqrt{1+4r^2\hat{\phi}^2 e^{2\kappa}}} \\
&-2\hat{h}^{ab} \tfrac{\hat{D}_a(r\hat{\phi}e^{\kappa})\hat{D}_b(r\hat{\phi}e^{\kappa})}{1+4r^2\hat{\phi}^2 e^{2\kappa}} - 4\hat{h}^{ab} \tfrac{\hat{D}_a(r\hat{\phi}e^{\kappa})\hat{D}_b \ln(r^2 e^{2\kappa})}{\sqrt{1+4r^2\hat{\phi}^2 e^{2\kappa}}} \\
&-2\hat{h}^{ab}\hat{D}_a \ln(r^2 e^{2\kappa})\hat{D}_b \ln(r^2 e^{2\kappa}) = 0.
\end{aligned} \tag{9}$$

To continue, we need some relations which hold in our specialized coordinate system. Each of the following statements are straightforward to check.

$$\begin{aligned}
&\hat{D}_i\hat{D}_j r^2 = 2\hat{h}_{ij} + D(\hat{h}_{ij}), \qquad \Delta r^2 = \hat{h}^{ij}\hat{D}_i\hat{D}_j r^2 = 6 + \hat{h}^{ij}D(\hat{h}_{ij}), \\
&\hat{h}^{ij}D(\hat{h}_{ij}) = D(\ln|\hat{h}|) \equiv 0 \pmod{r^2}, \qquad \forall f : \hat{h}^{jij}\hat{D}_i r^2\hat{D}_j f = 2D(f), \\
&\hat{h}^{ij}\hat{D}_i r\hat{D}_j r = 1.
\end{aligned} \tag{10}$$

We will now split (8) and (9) into their regular and non-regular parts. By a regular function we mean a function $f = f(x, y, z)$ such that $r^{2n}f$ is (formally) real analytic for some integer $n \geq 0$. Note that if $n \geq 1$ is required, the regular function is singular. Similarly, a function f is non-regular if $r^{2n-1}f$ is (formally) real analytic for some integer $n \geq 0$. Again, if $n \geq 1$ is required, the function is non-regular and singular. This division is due to the non-regularity of λ, and all functions or tensor fields can be written as $f = f_1 + rf_2$ where f_1 and f_2 are regular. (Cf. Lemma 2 of [8].) In particular, $f = 0$ requires $f_1 = f_2 = 0$. We note that $\hat{D}_a r = \frac{1}{2r}\hat{D}_a r^2$, which is non-regular.

On the other hand, $\hat{D}_a(r\hat{\phi}e^\kappa)\hat{D}_b(r\hat{\phi}e^\kappa) = [\frac{1}{2r}\hat{D}_a(r^2)\hat{\phi}e^\kappa + r\hat{D}_a(\hat{\phi}e^\kappa)][\frac{1}{2r}\hat{D}_b(r^2)\hat{\phi}e^\kappa + r\hat{D}_b(\hat{\phi}e^\kappa)] = \frac{\hat{\phi}^2e^{2\kappa}}{4r^2}\hat{D}_a r^2\hat{D}_b r^2 + r^2\hat{D}_a(\hat{\phi}e^\kappa)\hat{D}_b(\hat{\phi}e^\kappa) + \frac{1}{2}\hat{D}_{(a}(\hat{\phi}e^\kappa)\hat{D}_{b)}(r^2)\hat{\phi}e^\kappa$ which is regular (but singular). Therefore, the regular part of (8) is

$$\begin{aligned}&\hat{R}_{ab} + \hat{D}_a\hat{D}_b\ln(r^2e^{2\kappa}) + \hat{D}_a\ln(r^2e^{2\kappa})\hat{D}_b\ln(r^2e^{2\kappa})\\ &+\hat{h}_{ab}\hat{h}^{de}\hat{D}_d\hat{D}_e\ln(r^2e^{2\kappa}) - \hat{h}_{ab}\hat{h}^{de}\hat{D}_d\ln(r^2e^{2\kappa})\hat{D}_e\ln(r^2e^{2\kappa})\\ &-2\tfrac{\hat{D}_a(r\hat{\phi}e^\kappa)\hat{D}_b(r\hat{\phi}e^\kappa)}{1+4r^2\hat{\phi}^2e^{2\kappa}} = 0.\end{aligned} \tag{11}$$

while the non-regular part gives

$$\hat{h}_{ab}\hat{h}^{de}\hat{D}_d\frac{\hat{D}_e(r\hat{\phi}e^\kappa)}{\sqrt{1+4r^2\hat{\phi}^2e^{2\kappa}}} - \hat{h}_{ab}\hat{h}^{de}\frac{\hat{D}_d(r\hat{\phi}e^\kappa)\hat{D}_e\ln(r^2e^{2\kappa})}{\sqrt{1+4r^2\hat{\phi}^2e^{2\kappa}}} = 0. \tag{12}$$

Similarly, (9) splits into the two equations

$$\begin{aligned}&\hat{R} + 4\hat{h}^{ab}\hat{D}_a\hat{D}_b\ln(r^2e^{2\kappa}) - 2\hat{h}^{ab}\tfrac{\hat{D}_a(r\hat{\phi}e^\kappa)\hat{D}_b(r\hat{\phi}e^\kappa)}{1+4r^2\hat{\phi}^2e^{2\kappa}}\\ &-2\hat{h}^{ab}\hat{D}_a\ln(r^2e^{2\kappa})\hat{D}_b\ln(r^2e^{2\kappa}) = 0\end{aligned} \tag{13}$$

and (dividing by 4)

$$\hat{h}^{ab}\hat{D}_a\frac{\hat{D}_b(r\hat{\phi}e^\kappa)}{\sqrt{1+4r^2\hat{\phi}^2e^{2\kappa}}} - \hat{h}^{ab}\frac{\hat{D}_a(r\hat{\phi}e^\kappa)\hat{D}_b\ln(r^2e^{2\kappa})}{\sqrt{1+4r^2\hat{\phi}^2e^{2\kappa}}} = 0. \tag{14}$$

Equations (12) and (13) are redundant, since (12) is a multiple of (14), while (13) is the trace of (11). Since (14) is non-regular, $r\cdot$(14) is regular, and by using $r\hat{D}_i\hat{D}_j r = \frac{1}{2}\hat{D}_i\hat{D}_j r^2 - \frac{1}{4r^2}D_i r^2 D_j r^2$, it is also seen that $r\cdot$(14) is smooth at i^0. We therefore look at the singular part⁺ of (11), which is found to be

$$\begin{aligned}&\tfrac{1}{r^2}\hat{D}_i\hat{D}_j r^2 + \tfrac{2}{r^2}\hat{D}_i r^2\hat{D}_j\kappa + \tfrac{2}{r^2}\hat{D}_i\kappa\hat{D}_j r^2\\ &+\hat{h}_{ij}[\tfrac{1}{r^2}\Delta r^2 - \tfrac{8}{r^2} - \tfrac{8}{r^2}D(\kappa)] - \tfrac{\hat{\phi}^2e^{2\kappa}}{2r^2}\hat{D}_i r^2\hat{D}_j r^2.\end{aligned} \tag{15}$$

This expression must be smooth, and by taking its trace, this says that $\frac{4}{r^2}\Delta r^2 - \frac{24}{r^2} - \frac{16}{r^2}D(\kappa)$ must be smooth. However, from (10), this implies that $\frac{D(\kappa)}{r^2}$ is smooth, and therefore that $\kappa = C + r^2\chi$, where C is a constant, and χ is smooth. From $\kappa(i^0) = 0$ we thus infer that

$$\kappa = r^2\chi \tag{16}$$

for some smooth function χ. Inserting (16) in (15), some simplification shows that

$$\frac{1}{r^2}[2D(\hat{h}_{ij}) + \hat{D}_i r^2\hat{D}_j r^2(8\chi - \hat{\phi}^2)] \tag{17}$$

must be smooth. This equation, as it stands, has many solutions, and we must choose a solution which still allows the metric $\hat{h}_{ij}$ and the function $\kappa = r^2\chi$ to solve the equations (11) and (14). To do this, we start by imposing slightly more conditions on $\hat{h}_{ij}$, so that $\hat{h}_{ij}$ will still satisfy the conditions of Lemma 5, but in a slightly restricted form. Let A_1 be the matrix from Lemma 4. We will then require that $\hat{h}_{ij}$ takes the form

$$\hat{h}_{ij} = \eta_{ij} + f(x,y,z)(A_1)_{ij} + r^2\gamma_{ij} \tag{18}$$

⁺ In practise, skipping smooth terms

Since we have the factor r^2 explicitly in front of γ_{ij}, we only need to ensure that $\gamma_{ij}x^i = 0$ in order for $\hat{h}_{ij}$ to still satisfy the conditions of Lemma 5. This is guaranteed by the following lemma.

Lemma 9. *Suppose that γ_{ij} is such that $\gamma_{ij}x^i = 0$. Then γ_{ij} is uniquely decomposable as*

$$\begin{aligned}\gamma_{ij} = \gamma &= f_1(x,y)B_1 + f_2(x,y)B_2 + f_3(x,y)B_3 + f_4(x,y,z)B_4 \\ &\quad + f_5(x,y,z)B_5 + f_6(x,y,z)B_6, \text{ where}\end{aligned}$$

$$\begin{aligned}
B_1 &= \begin{pmatrix} -y^2 & xy & 0 \\ xy & -x^2 & \\ 0 & 0 & 0 \end{pmatrix} \\
B_2 &= \begin{pmatrix} 2yz & -xz & -xy \\ -xz & 0 & x^2 \\ -xy & x^2 & 0 \end{pmatrix} \\
B_3 &= \begin{pmatrix} 0 & -yz & y^2 \\ -yz & 2xz & -xy \\ y^2 & -xy & 0 \end{pmatrix} \\
B_4 &= \begin{pmatrix} z^2 & 0 & -xz \\ 0 & 0 & 0 \\ -xz & 0 & x^2 \end{pmatrix} \\
B_5 &= \begin{pmatrix} 0 & z^2 & -yz \\ z^2 & 0 & -xz \\ -yz & -xz & 2xy \end{pmatrix} \\
B_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & z^2 & -yz \\ 0 & -yz & y^2 \end{pmatrix}
\end{aligned}$$

Proof. This proof is rather similar to the proof of Lemma 4 and is given in appendix A. □

Now, $(A_1)_{ij} = x_i x_j - r^2\eta_{ij}$, and $4D(fx_ix_j) = (2f + D(f))\hat{D}_i r^2 \hat{D}_j r^2$. Using this, insertion of (18) into (17) then leaves us with the condition that

$$\frac{\hat{D}_i r^2 \hat{D}_j r^2}{r^2}[2f + D(f) + 2(8\chi - \hat{\phi}^2)] \qquad (19)$$

must be smooth at $r = 0$.

It is not trivial to impose the right conditions on f and χ. If they are chosen too restrictively, no solution to (11) and (14) will exist. On the other hand, if f and χ are not restricted enough, the solution we are looking for will not be unique (in terms of the introduced quantities). As we will see, the following choice, upon which we insist, will suffice.

$$\chi = \frac{1}{16}(2\hat{\phi}^2 - 2f - D(f)) \qquad (20)$$

The choice (20) will make (19) vanish identically; this means that the equations (11) and (14) now are smooth at i^0. Also, the form of the metric is determined via Lemma 9, with $f(x, y, z), f_1(x, y), f_2(x, y), f_3(x, y), f_4(x, y, z), f_5(x, y, z)$ and $f_6(x, y, z)$ as unknowns. We put $f_0 = f$ and note that with $\hat{h}_{ij}$ known the conformal factor Ω, and thus the desired spacetime, is also determined.

3.6. Uniqueness of the metric

We will now address the equations (11) and (14). We will first prove that a part of these equations determine the metric uniquely provided $\hat{h}_{ij}$ is cast in a certain way (Lemma 10), and then, in Section 3.7, that the derived series expansions for $\hat{h}_{ij}$ converges and also satisfies (11) and (14) in full (Lemma 13).

To prepare for Lemma 10 and Lemma 13, we will create some scalar equations from (11). To do this, we introduce the vectors fields u_1^a, u_2^a, u_3^a with components

$$u_1^i = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad u_2^i = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}, \quad u_3^i = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}$$

where it is seen that they are all pointwise orthogonal to the vector field x^i. We also note the linear relation

$$zu_1^i + xu_2^i + yu_3^i = 0.$$

It is straightforward to check that $u_1^i u_1^j, u_2^i u_2^j, u_3^i u_3^j, x^i x^j$ together with any two of $u_1^{(i} x^{j)}, u_2^{(i} x^{j)}, u_3^{(i} x^{j)}$ are (point-wise) lineareley independent Next, denote the LHS of (11) by T_{ab}, and denote $r\sqrt{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}}$ times the LHS of (14) by S. With the notation

$$t_{11} = u_1^i u_1^j T_{ij}, \quad t_{22} = u_2^i u_2^j T_{ij}, \quad t_{33} = u_3^i u_3^j T_{ij}$$

$$t_{00} = x^i x^j T_{ij}, \quad t_{0k} = x^i u_k^j T_{ij}, k = 1, 2, 3,$$

it is clear that (11) is satisfied if and only if $t_{00}, t_{11}, t_{22}, t_{33}$ and any two of $t_{0k}, k = 1, 2, 3$ vanishes.

Lemma 10. *Suppose that the metric components $\hat{h}_{ij}$ takes the form*

$$\begin{aligned} \hat{h}_{ij} = \hat{h} = \eta + f_0(x, y, z)A_1 + r^2 \left[f_1(x, y)B_1 + f_2(x, y)B_2 + f_3(x, y)B_3 \right. \\ \left. + f_4(x, y, z)B_4 + f_5(x, y, z)B_5 + f_6(x, y, z)B_6 \right], \end{aligned} \tag{21}$$

where A_1 and $B_1, B_2, \cdots B_6$ are the matrices in Lemmas 4 *and* 9 *respectively, and where the functions $f_0, f_1, \cdots f_6$ are formal analytical functions of the variables indicated. Then the equations*

$$S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0$$

determines the metric $\hat{h}_{ij}$ as formal power series.

This will be proved by induction. With the notation from Section 3.2,

$$\hat{h}_{ij} = \eta_{ij} + \sum_{n=2}^{\infty} \hat{h}_{ij}^{[n]}, \quad \hat{R}_{ij} = \sum_{n=0}^{\infty} \hat{R}_{ij}^{[n]}, \quad T_{ij} = \sum_{n=0}^{\infty} T_{ij}^{[n]}, \quad S = \sum_{n=0}^{\infty} S^{[n]}.$$

Note that there can be no linear term in the metric corresponding to $n = 1$ due to Lemma 4. Moreover, $\hat{h}_{ij}$ is determined by the functions $f_0, \cdots f_6$ with corresponding series

$$\begin{array}{rcl} f_k(x,y,z) &=& \sum_{n=0}^{\infty} f_k^{[n]}(x,y,z),\ k = 0,4,5,6, \\ f_k(x,y) &=& \sum_{n=0}^{\infty} f_k^{[n]}(x,y),\ k = 1,2,3, \end{array}$$

where each $f_k^{[n]}$ is a homogeneous polynomial (in the variables indicated) of degree n. We also recollect that κ is determined by f_0 and $\hat{\phi}$ via (16) and (20).

$\hat{R}_{ij}^{[n]}$ is determined by the metric up to order $n+2$, and with $\hat{h}_{ij}^{[n+2]}$ as the leading term, while $\hat{h}_{ij}^{[k]}, k \le n+1$ are regarded as lower order terms (L.O.T.) , the definition of the Ricci tensor gives that

$$\begin{array}{rcl} \hat{R}_{ij}^{[n]} &=& \underline{\hat{R}}_{ij}^{[n]} + \text{L.O.T.} \\ \underline{\hat{R}}_{ij}^{[n]} &=& \frac{1}{2}\eta^{mn}\partial_m \left\{ \partial_j \hat{h}_{in}^{[n+2]} + \partial_i \hat{h}_{jn}^{[n+2]} - \partial_n \hat{h}_{ij}^{[n+2]} \right\} \\ &-& \frac{1}{2}\eta^{mn}\partial_i \left\{ \partial_m \hat{h}_{jn}^{[n+2]} + \partial_j \hat{h}_{mn}^{[n+2]} - \partial_n \hat{h}_{mj}^{[n+2]} \right\} \end{array} \tag{22}$$

We stress that the lower order terms in $\hat{R}_{ij}^{[n]}$ are also polynomials of degree n, but that they are expressions in $\hat{h}_{ij}^{[k]}, k \le n+1$. Considering the form of $\hat{h}_{ij}$, $\underline{\hat{R}}_{ij}^{[n]}$ is a function of the leading order polynomials $f_0^{[n]}$, $f_k^{[n-2]}$ for $1 \le k \le 6$. We now proceed and look at T_{ij} and see how the leading order polynomials enter. From (11) we find

$$\begin{array}{l} (\hat{D}_i \hat{D}_j \ln(r^2))^{[n]} = (\frac{1}{r^2}\hat{D}_i\hat{D}_j r^2 - \frac{1}{r^4}\hat{D}_i r^2 \hat{D}_j r^2)^{[n]} \\ = \frac{1}{r^2}(2\hat{h}_{ij}^{[n+2]} + D(\hat{h}_{ij}^{[n+2]})) + \text{L.O.T.} = \frac{n+4}{r^2}\hat{h}_{ij}^{[n+2]} + \text{L.O.T.} \end{array} \tag{23}$$

Here we have used that $D(f) = nf$ if f is homogeneous of order n. Continuing, (11) gives

$$\begin{array}{rcl} 2(\hat{D}_i\hat{D}_j\kappa)^{[n]} &=& (\frac{1}{8}\hat{D}_i\hat{D}_j(r^2(2\hat{\phi}^2 - 2f - D(f))))^{[n]} \\ &=& \frac{-(2+n)}{8}\partial_i\partial_j(r^2 f_0^{[n]}) + \text{L.O.T.} \\ (\hat{D}_i \ln(r^2e^{2\kappa})\hat{D}_j \ln(r^2e^{2\kappa}))^{[n]} &=& \frac{-(n+2)}{4r^2}\partial_{(i} r^2 \partial_{j)}(r^2 f_0^{[n]}) + \text{L.O.T.} \\ (\hat{h}_{ij}\hat{h}^{nm}\hat{D}_n\hat{D}_m \ln(r^2))^{[n]} &=& \frac{n+2}{r^2}\eta_{ij}\eta^{nm}\hat{h}_{nm}^{[n+2]} + \frac{2}{r^2}\hat{h}_{nm}^{[n+2]} + \text{L.O.T.} \\ (2\hat{h}_{ij}\hat{h}^{nm}\hat{D}_n\hat{D}_m\kappa)^{[n]} &=& \frac{-(2+n)}{8}\eta_{ij}\eta^{nm}\partial_n\partial_m(r^2 f_0^{[n]}) + \text{L.O.T.} \\ (\hat{h}_{ij}\hat{h}^{nm}\hat{D}_n \ln(r^2e^{2\kappa})\hat{D}_m \ln(r^2e^{2\kappa}))^{[n]} &=& -\eta_{ij}\frac{(n+2)^2}{2}f_0^{[n]} + \frac{4}{r^2}\hat{h}_{nm}^{[n+2]} + \text{L.O.T.} \\ \left(2\frac{\hat{D}_i(r\hat{\phi}e^{\kappa})\hat{D}_j(r\hat{\phi}e^{\kappa})}{1+4r^2\hat{\phi}^2e^{2\kappa}}\right)^{[n]} &=& 0 + \text{L.O.T.} \end{array} \tag{24}$$

Combining (22), (23) and (24), we find the leading order of $T_{ij}^{[n]} = \underline{T}_{ij}^{[n]} +$ L.O.T. to be

$$\begin{array}{rl} \underline{T}_{ij}^{[n]} &= \frac{1}{2}\eta^{mn}\partial_m \left\{ \partial_j \hat{h}_{in}^{[n+2]} - \partial_n \hat{h}_{ij}^{[n+2]} \right\} - \frac{1}{2}\eta^{mn}\partial_i \left\{ \partial_j \hat{h}_{mn}^{[n+2]} - \partial_n \hat{h}_{mj}^{[n+2]} \right\} \\ & + \frac{n+2}{r^2}\hat{h}_{ij}^{[n+2]} + \frac{n+2}{r^2}\eta_{ij}\eta^{nm}\hat{h}_{nm}^{[n+2]} + \eta_{ij}\frac{(n+2)^2}{2}f_0^{[n]} \\ & - \frac{(n+2)}{8}[\partial_i\partial_j(r^2 f_0^{[n]}) + \frac{2}{r^2}\partial_{(i}r^2\partial_{j)}(r^2 f_0^{[n]}) + \eta_{ij}\eta^{nm}\partial_n\partial_m(r^2 f_0^{[n]})] \end{array}$$

To examine S, we write $Y = 2\hat{\phi}e^{\kappa}$, which means that $S/\sqrt{1+4r^2\hat{\phi}^2e^{2\kappa}}$ becomes

$$\begin{array}{l}\hat{h}^{ij} r\hat{D}_i\frac{\hat{D}_j(rY)}{\sqrt{1+r^2Y^2}} - \hat{h}^{ij}\frac{r\hat{D}_i(rY)\hat{D}_j\ln(r^2e^{2\kappa})}{\sqrt{1+r^2Y^2}}\\ = -\hat{h}^{ij}\frac{r^2Y\hat{D}_i(rY)\hat{D}_j(rY)}{\sqrt{1+r^2Y^2}^3} + \hat{h}^{ij}\frac{r\hat{D}_i\hat{D}_j(rY)}{\sqrt{1+r^2Y^2}} - \hat{h}^{ij}\frac{r\hat{D}_i(rY)[\hat{D}_j\ln(r^2)+2\hat{D}_j\kappa]}{\sqrt{1+r^2Y^2}}\end{array}$$

Expanding the derivatives and using (10), we find

$$\begin{array}{ll} S & = \frac{Y}{2}D(\ln|\hat{h}|) + r^2\Delta Y - 2YD(\kappa) - 2r^2\hat{h}^{ij}\hat{D}_iY\hat{D}_j\kappa \\ & -\frac{r^2Y}{1+r^2Y^2}[Y^2 + 2YD(Y) + r^2\hat{h}^{ij}\hat{D}_iY\hat{D}_j] \end{array} \tag{25}$$

Note that $S \equiv 0 \pmod{r^2}$. Now, using that $(\ln|\hat{h}|)^{[n+2]} = [\hat{h}^{[n+2]}_{ij}]$ + L.O.T., and putting $\Delta_C = \eta^{ij}\partial_i\partial_j$, it follows that (at this stage, $\hat{\phi}(0) \neq 0$ by assumption) $S^{[n+2]} = \underline{S}^{[n+2]}$+L.O.T., where

$$\begin{array}{ll} \underline{S}^{[n+2]} & = \frac{\hat{\phi}(0)}{2}((n+2)[\hat{h}^{[n+2]}_{ij}] + 2r^2\Delta_c\kappa^{[n+2]} - 4D(\kappa^{[n+2]})) \\ & = \frac{\hat{\phi}(0)(n+2)}{2}([\hat{h}^{[n+2]}_{ij}] - \frac{1}{8}r^2\Delta_c(r^2f^{[n]}_0) + \frac{1}{4}D(r^2f^{[n]}_0)). \end{array} \tag{26}$$

The leading order terms in (22), (24), (26) are all functions of the homogeneous polynomials $f^{[n]}_0$ and $f^{[n-2]}_k, k = 1, 2, \cdots 6$. $f^{[n]}_0(x, y, z)$ defines a vector space with dimension $\frac{(n+2)(n+1)}{2}$, while $f^{[n]}_k(x, y)$ $(k = 1, 2, 3)$ define a vector space with dimension $n + 1$. In total, the tuple (for each fixed n)

$$\overline{p} = (f^{[n]}_0, f^{[n-2]}_1, f^{[n-2]}_2, f^{[n-2]}_3, f^{[n-2]}_4, f^{[n-2]}_5, f^{[n-2]}_6)$$

lives in a vector space with dimension $2n^2 + 3n - 2$ for $n \geq 2$. On the other hand, $T^{[n]}_{ij}$ will show up in $t^{[n+2]}_{11}, t^{[n+2]}_{22}, t^{[n+2]}_{33}$, which are all polynomials of degree $\frac{(n+4)(n+3)}{2}$, so that the quadruple

$$\overline{t} = (S^{[n+2]}, t^{[n+2]}_{11}, t^{[n+2]}_{22}, t^{[n+2]}_{33})$$

lives in $\mathbf{R}^{2n^2+14n+24}$.

Thus, for each fixed n, $\overline{t}$ is a function of $\overline{p}$ and the functions $f^{[\cdot]}_i$ of lower order (than in $\overline{p}$). One also notes that both $(S^{[0]}, t^{[0]}_{11}, t^{[0]}_{22}, t^{[0]}_{33})$ and $(S^{[1]}, t^{[1]}_{11}, t^{[1]}_{22}, t^{[1]}_{33})$ are identically zero (see for instance Lemma 11 below). Next, it is easy to explicitly check that for $n = 0$, one can find $f^{[0]}_0$ so that $(S^{[2]}, t^{[2]}_{11}, t^{[2]}_{22}, t^{[2]}_{33}) = 0$ and similarly for $f^{[1]}_0$ when $n = 1$ (given $f^{[0]}_0$).

Using induction, we now assume that the metric $\hat{h}_{ij}$ is determined up to order $n + 1$, i.e., that the polynomials $f^{[m]}_0, m = 0, 1, \ldots n - 1$ and $f^{[m]}_k, k = 1, 2, \cdots 6, m = 0, 1, \ldots n - 3$ are known $(n \geq 2)$. With these polynomials fixed, $\overline{t} = \overline{t}(\overline{p})$ will be an affine function with respect to $\overline{p}$, and we must show that there exists a $\overline{p}$ such that $\overline{t}(\overline{p}) = 0$.

With $\overline{t}_0 = \overline{t}(0)$, we consider the equation

$$\overline{t}(\overline{p}) - \overline{t}_0 = -\overline{t}_0 \tag{27}$$

where now the mapping $\overline{q} : \overline{p} \mapsto \overline{q}(\overline{p}) = \overline{t}(\overline{p}) - \overline{t}_0$ is linear. Explicitly, $\overline{q}(\overline{p}) = (\underline{S}^{[n+2]}, u^i_1u^j_1\underline{T}^{[n]}_{ij}, u^i_2u^j_2\underline{T}^{[n]}_{ij}, u^i_3u^j_3\underline{T}^{[n]}_{ij})$. Since $\overline{q}$ is a mapping

$$\overline{q} : \mathbf{R}^{2n^2+3n-2} \to \mathbf{R}^{2n^2+14n+24},$$

we have a system of equations with $2n^2+14n+24$ equations and $2n^2+3n-2$ unknowns, i.e., the system is over-determined.

To prove that there is a $\overline{p}$ for which $\overline{q}(\overline{p}) = -\overline{t}_0$, we will prove i) that the dimension of the range of $\overline{t}$, and therefore $\overline{q}$, is only $2n^2+3n-2$, and ii) the mapping $\overline{p} \to \overline{q}(\overline{p})$ is injective.

To do this, we start with the somewhat surprising* lemma

Lemma 11. $S \equiv 0, (\ mod\ r^2), \quad t_{ii} \equiv 0, \quad (\mathrm{mod}\ r^2),\ i = 1,2,3$

Proof. From (25), using that $D(\ln|\hat{h}|) \equiv 0, \quad (\mathrm{mod}\ r^2)$, it immediately follows that $S \equiv 0, \quad (\mathrm{mod}\ r^2)$. The other properties are more tedious to prove, and we refer to appendix A for these calculations. □

Since all of $S, t_{11}, t_{22}, t_{33}$ contains the factor r^2, we can write

$$S = r^2\sigma, \quad t_{ii} = r^2\tau_{ii},\ i = 1,2,3.$$

As a consequence, $S^{[n+2]}$ is replaced by $\sigma^{[n]}$, and similarly, $t_{ii}^{[n+2]}$ is replaced by $\tau_{ii}^{[n]}$, and we consider the quadruple

$$\bar{\bar{t}} = (\sigma^{[n]}, \tau_{11}^{[n]}, \tau_{22}^{]n]}, \tau_{33}^{[n]}) \in \mathbf{R}^{2n^2+6n+4}$$

where we now have to solve the equation $\bar{\bar{t}}(\overline{p}) = 0$. As before, we can put $\bar{\bar{t}}_0 = \bar{\bar{t}}(0)$ $(=\overline{t}(0)/r^2)$ and address the equation $\bar{\bar{q}}(\overline{p}) = -\bar{\bar{t}}_0$ where $\bar{\bar{q}}$ is the linear function $\bar{\bar{q}}(\overline{p}) = \bar{\bar{t}}(\overline{p}) - \bar{\bar{t}}_0$, which is a mapping $\mathbf{R}^{2n^2+3n-2} \to \mathbf{R}^{2n^2+6n+4}$. To find a $\overline{p}$ so that $\bar{\bar{q}}(\overline{p}) = 0$ therefore gives $2n^2+6n+4$ equations in $2n^2+3n-2$ unknowns, i.e., the system of equations is still over-determined. However, there are additionally $3n+6$ linear relations among $(\tau_{11}^{[n]}, \tau_{22}^{[n]}, \tau_{33}^{[n]})$ as the following arguments show. First, Lemma 11 showed that

$$\begin{aligned} t_{11} &= y^2T_{11} - 2xyT_{12} + x^2T_{22} = (x^2+y^2+z^2)\tau_{11}(x,y,z) \\ t_{22} &= z^2T_{22} - 2yzT_{23} + y^2T_{33} = (x^2+y^2+z^2)\tau_{22}(x,y,z) \\ t_{33} &= x^2T_{33} - 2xzT_{13} + z^2T_{11} = (x^2+y^2+z^2)\tau_{33}(x,y,z) \end{aligned}$$

Putting $x = 0$, the first and third relation give

$$y^2T_{11}(0,y,z) = (y^2+z^2)\tau_{11}(0,y,z), \quad z^2T_{11}(0,y,z) = (y^2+z^2)\tau_{33}(0,y,z)$$

which implies

$$z^2\tau_{11}(0,y,z) = y^2\tau_{33}(0,y,z). \tag{28}$$

With the ansatz $\tau_{11}(0,y,z) = \sum_{k=0}^n a_{11,k}y^kz^{n-k}$, $\tau_{33}(0,y,z) = \sum_{k=0}^n a_{33,k}y^kz^{n-k}$, (28) implies that $a_{11,0} = a_{11,1} = 0$, $a_{11,k} = b_{33,k-2}, k = 2,3,\ldots n$, $a_{33,n-1} = a_{33,n} = 0$, i.e, $n+3$ linear relations. Putting $y = 0$ produces another $n+3$ linear relations, as does $z = 0$. However, it is easily seen that three relations are counted twice, which menas that the total number of linear relations are $3n+6$.

Thus, the range of $\bar{\bar{q}}$ is a vector space with dimension $2n^2+3n-2$ (in which $-\bar{\bar{t}}_0$ lives) and hence we have shown that the equation (27) has a solution if the mapping $\overline{q}$ is injective. This is the content of Lemma 12.

* For instance, $u_1^iu_1^j\hat{R}_{ij}$ is not congruent to 0 $\quad (\mathrm{mod}\ r^2)$.

Lemma 12. *Let the linear mapping* $\overline{q} : \mathbf{R}^{2n^2+3n-2} \to \mathbf{R}^{2n^2+6n+4}$ *be defined by* $\overline{q}(\overline{p}) = (\underline{S}^{[n+2]}, u_1^i u_1^j \underline{T}_{ij}^{[n]}, u_2^i u_2^j \underline{T}_{ij}^{[n]}, u_3^i u_3^j \underline{T}_{ij}^{[n]})$. *Then* $\overline{q}$ *is injective.*

Proof. Assume that $\overline{q}(\overline{p}) = 0$. Consider the matrix γ_{ij} in the decomposition (18) of $\hat{h}_{ij}$. By splitting♯ $\gamma_{ij} = \tilde{\gamma}_{ij} + \frac{1}{2}\gamma r^2 d\Omega_{ij}$ where $\tilde{\gamma}_{ij}$ is trace free, $\gamma = [\gamma_{ij}]$, and where $d\Omega_{ij}$ is the metric of the unit sphere S^2, we see that

$$\hat{h}_{ij}^{[n+2]} = f_0^{[n]}(x,y,z)(A_1)_{ij} + r^2\tilde{\gamma}_{ij}^{[n]} + r^2\frac{1}{2}\gamma^{[n]} r^2 d\Omega_{ij}$$

and in particular that

$$[\hat{h}_{ij}^{[n+2]}] = -2r^2 f_0^{[n]}(x,y,z) + r^2\gamma^{[n]}$$

The first 'component' in $\overline{q}(\overline{p})$ (i.e., $S^{[n+2]}$) being zero therefore implies

$$\gamma^{[n]} = \frac{1}{8}[12 f_0^{[n]} + \Delta_C(r^2 f_0^{[n]}) - 2D(f_0^{[n]})].$$

With the decomposition $\Delta_C = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2}\Delta_S$, where Δ_S is the angular Laplacian on the unit sphere, this can also be written

$$\gamma^{[n]} = \frac{1}{8}[(n^2+3n+18) f_0^{[n]} + \Delta_S f_0^{[n]}]. \tag{29}$$

We now express $\tilde{\gamma}_{ij}^{[n]}$ in terms of spherical coordinates (r,θ,ϕ), and get

$$\tilde{\gamma}_{ij}^{[n]} dx^i dx^j = (dr\ d\theta\ d\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & r^{n+2} f_{\tilde{\gamma}}(\theta,\phi) & r^{n+2} g_{\tilde{\gamma}}(\theta,\phi) \\ 0 & r^{n+2} g_{\tilde{\gamma}}(\theta,\phi) & -r^{n+2}\sin^2(\theta) f_{\tilde{\gamma}}(\theta,\phi) \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

for some angular functions $f_{\tilde{\gamma}}(\theta,\phi)$ and $g_{\tilde{\gamma}}(\theta,\phi)$. Note that $f_{\tilde{\gamma}}$ and $g_{\tilde{\gamma}}$ are not arbitrary since $\tilde{\gamma}_{ij}$ must be analytic when written in the coordinates (x,y,z). With this representation, $u_k^i u_k^j \underline{T}_{ij}^{[n]}$ $(k = 1,2,3)$ are functions of $f_0^{[n]}$, $\gamma^{[n]}$, $f_{\tilde{\gamma}}$ and $g_{\tilde{\gamma}}$.

In particular, $u_1^i u_1^j \underline{T}_{ij}^{[n]} = 0$ gives the equation

$$\begin{aligned} 4\sin^2\theta \left[2\,\tfrac{\partial^2}{\partial\theta^2} f_{\tilde{\gamma}} + (n+3)n\, f_{\tilde{\gamma}} - \Delta_S f_{\tilde{\gamma}}\right] + 16\,\sin\theta\cos\theta\tfrac{\partial}{\partial\theta} f_{\tilde{\gamma}} \\ +8\,\tfrac{\partial^2}{\partial\theta\partial\varphi} g_{\tilde{\gamma}} - 2\,\sin^2\theta\left(\Delta_S\gamma^{[n]} + n(n+1)\gamma^{[n]}\right) \\ -\sin^2\theta\left(2\,n\,\Delta_S f_0^{[n]} + n^3 f_0^{[n]} - (n+2)\,\tfrac{\partial^2}{\partial\theta^2} f_0^{[n]}\right) = 0. \end{aligned} \tag{30}$$

♯ This splitting is possible since $\gamma_{ij}x^j = 0$.

The equations $u_2^i u_2^j \underline{T}_{ij}^{[n]} = 0$ and $u_3^i u_3^j \underline{T}_{ij}^{[n]} = 0$ are slightly longer, namely

$$\begin{aligned}
4\,\mathcal{A}\left[\sin^2\theta\tfrac{\partial^2 f_{\tilde\gamma}}{\partial\theta^2} + 3\cos\theta\sin\theta\tfrac{\partial f_{\tilde\gamma}}{\partial\theta} - \tfrac{\partial^2 f_{\tilde\gamma}}{\partial\phi^2}\right] \\
+4\,\sin^2\theta((n^2+3n)\left[\mathcal{A} - 2\sin^2\phi\right] - 4\sin^2\phi)f_{\tilde\gamma} \\
+8\,\mathcal{A}\tfrac{\partial^2 g_{\tilde\gamma}}{\partial\theta\partial\phi} - 8(n^2+3n+2)\,\sin\theta\sin\phi\cos\phi\cos\theta g_{\tilde\gamma} \\
-2\mathcal{A}\left[\sin^2\theta\tfrac{\partial^2\gamma^{[n]}}{\partial\theta^2} + \sin\theta\cos\theta\tfrac{\partial\gamma^{[n]}}{\partial\theta} + \tfrac{\partial^2\gamma^{[n]}}{\partial\phi^2} + n\,(n+1)\sin^2\theta\gamma^{[n]}\right] \\
-2(n+2)\sin\theta\sin\phi\cos\phi\cos\theta\tfrac{\partial^2 f_0^{[n]}}{\partial\theta\partial\phi} \\
-\sin^2\theta\left((n-2)\mathcal{A} + (n+2)\sin^2\phi\right)\tfrac{\partial^2 f_0^{[n]}}{\partial\theta^2} \\
+\left((n+2)\sin^2\phi - 2n\mathcal{A}\right)\tfrac{\partial^2 f_0^{[n]}}{\partial\phi^2} + 2(n+2)\sin\phi\cos\phi\cos^2\theta\tfrac{\partial f_0^{[n]}}{\partial\phi} \\
-\sin\theta\cos\theta\left((n-2)\sin^2\phi + 2n\,\cos^2\phi\cos^2\theta\right)\tfrac{\partial f_0^{[n]}}{\partial\theta} - n^3\sin^2\theta\mathcal{A}f_0^{[n]} = 0, \\
\mathcal{A} = 1 - \cos^2\phi\sin^2\theta
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
4\mathcal{B}\left[\sin^2\theta\tfrac{\partial^2 f_{\tilde\gamma}}{\partial\theta^2} + 3\sin\theta\cos\theta\tfrac{\partial f_{\tilde\gamma}}{\partial\theta} - \tfrac{\partial^2 f_{\tilde\gamma}}{\partial\phi^2}\right] \\
+4\,\sin^2\theta((n^2+3n)\left[\mathcal{B} - 2\cos^2\phi\right] - 4\cos^2\phi)f_{\tilde\gamma} \\
8\mathcal{B}\tfrac{\partial^2 g_{\tilde\gamma}}{\partial\theta\partial\phi} + 8(n^2+3n+2)\,\sin\phi\cos\phi\cos\theta\sin\theta g_{\tilde\gamma} \\
-2\mathcal{B}\left[\sin^2\theta\tfrac{\partial^2\gamma^{[n]}}{\partial\theta^2} + \sin\theta\cos\theta\tfrac{\partial\gamma^{[n]}}{\partial\theta} + \tfrac{\partial^2\gamma^{[n]}}{\partial\phi^2} + n\,(n+1)\sin^2\theta\gamma^{[n]}\right] \\
+2(n+2)\sin\theta\sin\phi\cos\phi\cos\theta\tfrac{\partial^2 f_0^{[n]}}{\partial\theta\partial\phi} \\
-\sin^2\theta\left((n-2)\mathcal{B} + (n+2)\cos^2\phi\right)\tfrac{\partial^2 f_0^{[n]}}{\partial\theta^2} \\
+\left((n+2)\,\cos^2\phi - 2\,n\mathcal{B}\right)\tfrac{\partial^2 f_0^{[n]}}{\partial\phi^2} - 2(n+2)\sin\phi\cos\phi\cos^2\theta\tfrac{\partial f_0^{[n]}}{\partial\phi} \\
-\sin\theta\cos\theta\left((n-2)\cos^2\phi + 2n\sin^2\phi\cos^2\theta\right)\tfrac{\partial f_0^{[n]}}{\partial\theta} - n^3\sin^2\theta\mathcal{B}f_0^{[n]} = 0, \\
\mathcal{B} = 1 - \sin^2\phi\sin^2\theta
\end{aligned} \tag{32}$$

$\gamma^{[n]}$ can be eliminated using (29), and after insertion in (30), one can express $\frac{\partial^2 g_{\tilde\gamma}}{\partial\theta\partial\varphi}$ in terms of $f_0^{[n]}$ and $f_{\tilde\gamma}$. With $\gamma^{[n]}$ and $\frac{\partial^2 g_{\tilde\gamma}}{\partial\theta\partial\varphi}$ inserted in (31), one can 'solve' for $g_{\tilde\gamma}$, i.e., express also $g_{\tilde\gamma}$ in terms of $f_0^{[n]}$ and $f_{\tilde\gamma}$. From this, (32) gives $f_{\tilde\gamma}$, and therefore also $g_{\tilde\gamma}$ and $\gamma^{[n]}$ as a function of $f_0^{[n]}$. Finally, we insert all these quantities in $u_1^i u_1^j \underline{T}_{ij}^{[n]}$. Using the obtained expression for $\frac{\partial^2 g_{\tilde\gamma}}{\partial\theta\partial\varphi}$ gives a trivial identity, but by using the expression for $g_{\tilde\gamma}$, differentiated with respect to θ and φ, we get a fourth order linear equation for $f_0^{[n]}$. We now put $f_0^{[n]} = r^n f_a(\theta,\phi)$, which in particular means that $f_a(\theta,\phi)$ is a linear combination of spherical harmonics $Y_l^m(\theta,\phi)$ with $l \leq n$. After these steps, the equation for the angular part $f_a(\theta,\phi)$ is found to be

$$\begin{aligned}
\sin^2\theta\tfrac{\partial^4 f_a}{\partial\theta^4} + 2\,\tfrac{\partial^4 f_a}{\partial\theta^2\partial\varphi^2} + \frac{\frac{\partial^4 f_a}{\partial\varphi^4}}{\sin^2\theta} + 2\,\cos\theta\sin\theta\tfrac{\partial^3 f_a}{\partial\theta^3} - 2\,\frac{\cos\theta\,\frac{\partial^3 f_a}{\partial\theta\partial\varphi^2}}{\sin\theta} \\
-\left((2n^2+6n+5)\cos^2\theta - 2n^2 - 6n - 4\right)\tfrac{\partial^2 f_a}{\partial\theta^2} \\
-2\,\frac{\left(n^2+3n+2\right)\cos^2\theta - n^2 - 3n - 4\big)\frac{\partial^2 f_a}{\partial\varphi^2}}{\sin^2\theta} \\
-\frac{\cos\theta\left((2n^2+6n+6)\cos^2\theta - 2n^2 - 6n - 7\right)\frac{\partial f_a}{\partial\theta}}{\sin\theta} \\
+\sin^2\theta\,(n+3)\,(n+2)\,(n+1)\,n\,f_a = 0.
\end{aligned} \tag{33}$$

Since the coefficients of (33) do not contain ϕ, we make the ansatz $f_a(\theta,\phi) = F(\theta)e^{im\phi}$. The equation for $F(\theta)$ becomes

$$\begin{aligned}\sin^2\theta\tfrac{d^4}{d\theta^4}F(\theta) + 2\cos\theta\sin\theta\tfrac{d^3}{d\theta^3}F(\theta)\\ -[2m^2 + ((2n^2+6n+5)\cos^2\theta - 2n^2 - 6n - 4)]\tfrac{d^2}{d\theta^2}F(\theta)\\ +[2m^2 - ((2n^2+6n+6)\cos^2\theta - 2n^2 - 6n - 7)]\tfrac{\cos\theta}{\sin\theta}\tfrac{d}{d\theta}F(\theta)\\ +[\tfrac{m^4+2m^2\left((n^2+3n+2)\cos^2\theta - n^2 - 3n - 4\right)}{\sin^2\theta}\\ +\sin^2\theta(n+3)(n+2)(n+1)n]F(\theta) = 0\end{aligned} \tag{34}$$

The solution to (34) is

$$\begin{aligned}F(\theta) = C_1\sin\theta P_{n+1}^{m-1}(\cos\theta) + C_2\sin\theta P_{n+1}^{m+1}(\cos\theta)\\ +C_3\sin\theta Q_{n+1}^{m-1}(\cos\theta) + C_4\sin\theta Q_{n+1}^{m+1}(\cos\theta)\end{aligned} \tag{35}$$

i.e., a linear combination of associated Legendre functions of both kinds. Now, since $f_a(\theta,\phi)$ consists of spherical harmonics $Y_l^m(\theta,\phi)$ with $l \le n$, (35) implies that the relation $f_a(\theta,\phi) = F(\theta)e^{im\phi}$ requires that F and hence f_a is identically zero. From this, (29) gives $\gamma^{[n]} = 0$. By forming $(1+\cos^2\theta)u_1u_1\underline{T}_{ij}^{[n]} - (u_2u_2\underline{T}_{ij}^{[n]} + u_3u_3\underline{T}_{ij}^{[n]})\sin^2\theta$ it is seen that $f_{\tilde{\gamma}} = 0$, from which $g_{\tilde{\gamma}}$ must also be zero. This means that the mapping $\overline{q}$ is injective, and therefore that there exists a unique formal series expansion of $\hat{h}_{ij}$ such that $S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0$ (Lemma 10) □

We again stress that (27) gives the metric components of the appropriate order explicitly.

3.7. Convergence of the metric

In this section we will prove that the series expansion for $\hat{h}_{ij}$, which was found in the previous section, converges and also solves the full conformal field equations. This is the content of the following lemma.

Lemma 13. *Suppose that $\hat{h}_{ij}$ is a formal power series for the metric of the form* (21)*, producing the moments of Theorem* 1*. If*

$$S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0,$$

then

$$t_{00} = 0, \quad t_{0k} = 0, k = 1, 2, 3$$

and the power series is convergent in a neighbourhood of i^0.

This will conclude the proof of Theorem 1 when the monopole is nonzero.

Proof. To prove this lemma, we will use a result by Friedrich, namely Theorem 1.1 in [3], where a related property is investigated. In [3] it is proven that under rather similar settings, there exists a metric which instead of prescribed multipole moments, produces a set of prescribed *null data*, where also growth conditions for the null data in order for the series expansion of the metric to converge are given. These null data are proven to

be in a one-to-one correspondence with the family of multipole moments with non-zero monopole, but since this correspondence is rather implicit, the actual conditions on the multipole moments (as required by the conjecture by Geroch) are not clear.

We vill prove that the conditions on the moments $P^0, P^0_{i_1}, P^0_{i_1 i_2}, \cdots$ in Theorem 1 can be carried over to estimates on the null data as required in [3]. This will then guarantee a solution to the conformal Einstein's field equations which according to the work presented here will have the desired multipole moments.

In order to be able to refer to the work in [3], we continue to impose the temporary condition that the monopole P^0 is non-zero.

To begin, we compare the different conformal settings used. In [3], one uses the conformal factor

$$\Omega_F = \left(\frac{1-\sqrt{\lambda}}{m}\right)^2$$

while this work uses (starting with $(h_G)_{ab}$ in Section 3.3) the conformal factor

$$\hat{\Omega} = \sqrt{\lambda}\Omega = \sqrt{\lambda} r^2 e^{2\kappa}$$

Therefore, if $(h_F)_{ab}$ denotes the metric used in [3], we have the relation

$$(h_F)_{ab} = \left[\frac{1}{m^2\Omega}\frac{(1-\sqrt{\lambda})^2}{\sqrt{\lambda}}\right]^2 \hat{h}_{ab}, \quad \lambda = \sqrt{1+4\Omega\hat{\phi}^2} + 2\sqrt{\Omega}\hat{\phi}, \quad \Omega = r^2 e^{2\kappa}.$$

Since $\sqrt{\Omega}$ contains the non-smooth quantity r, it is not obvious that the transition from $\hat{h}_{ab}$ to $(h_F)_{ab}$ is smooth. However, by putting $\xi = 2re^{\kappa}\hat{\phi}$, the conformal factor Ω_T relating $\hat{h}_{ab}$ and $(h_F)_{ab}$ is seen to be

$$\Omega_T = \frac{1}{m^2\Omega}\frac{(1-\sqrt{\lambda})^2}{\sqrt{\lambda}} = \frac{4\hat{\phi}^2}{m^2}\frac{\left(1-\sqrt{\sqrt{1+\xi^2}+\xi}\right)^2}{\xi^2\sqrt{\sqrt{1+\xi^2}+\xi}}.$$

By using $\sqrt{1+\xi^2}-\xi = \frac{1}{\sqrt{1+\xi^2}+\xi}$, it follows that this expression is even in ξ, which means that only even powers of r occurs in Ω_T. Since the limit $\lim_{r\to 0}\Omega_T = 1$ causes no problem, Ω_T is found to be (formally) smooth. Note that since we know that $\hat{\phi}$ produces the prescribed moments under $\hat{h}_{ab}$, the potential $\hat{\phi}/\sqrt{\Omega_T}$ will give the correct moments when using $(h_F)_{ab}$.

Now, suppose that $S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0$, and hence, according to Lemma 10, that we have a formal metric $\hat{h}_{ij}$. Going over to $(h_F)_{ab}$, and using the trace-free Ricci tensor, this defines a set of abstract null data ($\mathcal{D}_n$ and $\mathcal{D}_n^*$ below). By the arguments in [3], there exists a formal solution to the full conformal static field equations connected to these null data, and by going back to $\hat{h}_{ij}$, this implies that also $t_{00} = 0, t_{0k} = 0, k = 1, 2, 3$.

Another way of putting this is: if $(h_F)_{ab}$ satisfies the conformal static field equations and if $(h_F)_{ab}$ and $\hat{h}_{ab}$ are conformally related, the equations (4) are automatically

satisfied, while the equations $S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0$ fully determines the metric components $\hat{h}_{ij}$ in terms of the normal coordinates x, y, z (taking the special form if $\hat{h}_{ij}$ into account).

In [3], one uses normal coordinates around i^0, and introduces a frame field $c_{\mathbf{a}}, \mathbf{a} = 1, 2, 3$ which is parallely propagated along the geodesics through i^0. Since we write $\hat{R}_{ij}$ for the Ricci tensor, we can let R_{ab} stand for the Ricci tensor in [3]. With s_{ab} denoting the trace free part of the Ricci tensor, we have $s_{ab} = R_{ab}$, since R_{ab} in [3] already is trace free due to the choice of conformal gauge there. One then consider the set

$$\mathcal{D}_n = \{s_{a_2a_1}(i^0), C[\nabla_{a_3}s_{a_2a_1}](i^0), C[\nabla_{a_4}\nabla_{a_3}s_{a_2a_1}](i^0), \ldots\}$$

where ∇_a is the derivative operator associated with $(h_F)_{ab}$. To proceed, one then expresses the family of tensors in $\mathcal{D}_n$ in the introduced frame field, and considers (again, see [3] for the details) the related family

$$\mathcal{D}_n^* = \{s_{\mathbf{a_2a_1}}(i^0), C[\nabla_{\mathbf{a_3}}s_{\mathbf{a_2a_1}}](i^0), C[\nabla_{\mathbf{a_4}}\nabla_{\mathbf{a_3}}s_{\mathbf{a_2a_1}}](i^0), \ldots\}$$

According to Theorem 1.1 in [3], there exists an analytic solution around i^0 to the conformal static vacuum field equations with $m \neq 0$ for the conformal metric $(h_F)_{ab}$ to each set of tensors, given in the orthonormal frame (at i^0),

$$\hat{\mathcal{D}}_n = \{\psi_{\mathbf{a_2a_1}}, \psi_{\mathbf{a_3a_2a_1}}, \psi_{\mathbf{a_4a_3a_2a_1}}, \ldots\}$$

which satisfy: i) each tensor is totally symmetric and trace free, and ii) there exist constants $M, r > 0$ such that

$$|\psi_{\mathbf{a_p\ldots a_1bc}}| \leq \frac{Mp!}{r^p}, \quad a_p, \ldots a_1, b, c = 1, 2, 3, \quad p = 0, 1, 2, \ldots \tag{36}$$

In particular, in the introduced frame, one has

$$C[\nabla_{\mathbf{a_q}} \ldots \nabla_{\mathbf{a_3}}s_{\mathbf{a_2a_1}}](i^0) = \psi_{\mathbf{a_q\ldots a_1}}, \quad q = 2, 3, 4, \ldots \tag{37}$$

We will show that given the conditions of Theorem 1, there will exist such a family $\hat{\mathcal{D}}_n$ with the desired properties and, most importantly, that the tensors in $\hat{\mathcal{D}}_n$ produce the prescribed moments via the requirement (37). We have already concluded that given $\hat{h}_{ij}$, Ω_T and $\hat{\phi}$, this defines $(h_F)_{ab}$, and moreover that $\hat{\phi}/\sqrt{\Omega_T}$ will give the correct moments when using $(h_F)_{ab}$. We can therefore define $\psi_{\mathbf{a_q\ldots a_1}}$ in (37) as the corresponding left hand side and prove the necessary estimates on $C[\nabla_{\mathbf{a_q}} \ldots \nabla_{\mathbf{a_3}}s_{\mathbf{a_2a_1}}](i^0)$ directly.

Now, with the notation in Theorem 1, the convergence of $\sum_{|\alpha|\geq 0} \frac{\mathbf{r}^\alpha}{\alpha!}P_\alpha^0$ near i^0 implies that for some constants $M, r > 0$,

$$|P_\alpha^0| \leq \frac{p!M}{r^p}, \quad p = 0, 1, 2, \ldots \tag{38}$$

From the arrangement in [3], $P_{a_1a_2} = -\frac{m}{2}s_{a_1a_2}$, and hence the estimates in (36) will be satisfied if there are constants $\hat{M}, \hat{r}$ so that

$$|C[\nabla_{\mathbf{a_q}} \ldots \nabla_{\mathbf{a_3}}P_{\mathbf{a_2a_1}}](i^0)| \leq \frac{|\alpha|!\hat{M}}{\hat{r}^{|\alpha|}}, \quad |\alpha| = p = 0, 1, 2, \ldots$$

Proceeding as in [3], the tensors P_α can be expressed as totally symmetric space spinors fulfilling a certain reality condition [(3.4) in [3]]. In terms of these space spinors, (38) reads

$$|P^0_{A_pB_p\dots A_1B_1}| \le \frac{p!\tilde{M}}{\tilde{r}^p}, \quad A_p, B_p, \dots A_1, B_1 = 0, 1, \quad p = 0, 1, 2, \dots \tag{39}$$

for some related constants $\tilde{M}, \tilde{r}$, and we must demonstrate that given the estimates (39), there are $\check{M}, \check{r}$ so that

$$|D_{(A_pB_p} \dots D_{A_3B_3} P_{A_2B_2A_1B_1)}(i^0)| \le \frac{p!\check{M}}{\check{r}^p}, \ A_p, B_p, \dots A_1, B_1 = 0, 1, \ p = 0, 1, \dots$$

This will be done using induction, and to get the arguments to work, we first observe that given (39), it is easy to find constants M_0, r_0 so that

$$|P_{A_pB_p\dots A_1B_1}(i^0)| \le \frac{(p-1)!M_0}{r_0^p}, \quad A_p, B_p, \dots A_1, B_1 = 0, 1, \quad p = 1, 2, 3 \dots \tag{40}$$

To simplify the calculations, we introduce the following notation. We denote $P_{A_pB_p\dots A_1B_1}$ by P_p, and let $D_{(A_pB_p} \dots D_{A_{k+1}B_{k+1}} P_{A_kB_k\dots A_2B_2A_1B_1)}$ be denoted by $D_{p-k}P_k$. In terms of this notation, the recursion (2) expressed in space spinors becomes ($c_p = \frac{p(2p-1)}{2}$)

$$P_p = D_1 P_{p-1} - c_{p-1} R_2 P_{p-2} \tag{41}$$

where R_2 stands for $R_{A_1B_1A_2B_2}$ and it is understood that all products involves complete symmetrization. For instance, R_2P_{p-2} stands for $R_{(A_pB_pA_{p-1}B_{p-1}} P_{A_{p-2}B_{p-2}\dots A_2B_2A_1B_1)}$. Since $P_{ab} = -\frac{m}{2}R_{ab}$, a simple rescaling $P_{a_p\dots a_1} \to -\frac{m}{2}P_{a_p\dots a_1}$ (keeping the same stem letter), transforms (41) into

$$D_1 P_p = P_{p+1} + c_p P_{p-1} P_2$$

We now claim that

$$\forall n \ge 0, |D_n P_p(i^0)| \le \frac{M_0(1+2M_0)^n (n+p-1)!}{r_0^{n+p}}, \quad p = 2, 3, 4, \dots \tag{42}$$

where (40) is the initial estimate for $n = 0$, and where we assume that (42) is valid up to a certain value of n. For $n+1$ we then get

$$D_{n+1}P_p = D_n P_{p+1} + c_p \sum_{k=0}^{n} \binom{n}{k} D_k P_{p-1} D_{n-k} P_2,$$

and inserting the estimimates (42) we get, where (43) to (46) are evaluated at (i^0),

$$\begin{aligned} |D_{n+1}P_p| &\le |D_nP_{p+1}| + c_p \textstyle\sum_{k=0}^n \binom{n}{k} |D_kP_{p-1}||D_{n-k}P_2| \\ &\le \frac{M_0(1+2M_0)^n(n+p)!}{r_0^{n+p+1}} + c_p \textstyle\sum_{k=0}^n \binom{n}{k} \frac{M_0^2(1+2M_0)^n(k+p-2)!(n-k+1)!}{r_0^{n+p+1}}. \end{aligned} \tag{43}$$

Next, from $c_p \le 2p(p-1)$, for $p \ge 2$, we get

$$|D_{n+1}P_p| \le \frac{M_0(1+2M_0)^n}{r_0^{n+p+1}} \left[(n+p)! + 2M_0 p(p-1) \sum_{k=0}^{n} \binom{n}{k} (k+p-2)!(n-k+1)! \right] \tag{44}$$

However, from the identity

$$p(p-1)\sum_{k=0}^{n}\binom{n}{k}(k+p-2)!(n-k+1)!=(n+p)!,$$

we get

$$|D_{n+1}P_p|\leq\frac{M_0(1+2M_0)^n}{r_0^{n+p+1}}\left[(n+p)!+2M_0(n+p)!\right]=\frac{M_0(1+2M_0)^{n+1}(n+p)!}{r_0^{n+p+1}}\quad(45)$$

as claimed in (42). Thus the estimates in (42) are valid, and by putting $p=2$, we find that (at i^0)

$$\forall n\geq 0,|D_nP_2|\leq\frac{M_0(1+2M_0)^n(n+1)!}{r_0^{n+2}}\quad(46)$$

Since it is easy to find constants $\check{M},\check{r}$ such that $\frac{M_0(1+2M_0)^n(n+1)!}{r_0^{n+2}}\leq\frac{n!\check{M}}{\check{r}^n}$ for $n=0,1,2,\cdots$, Theorem 1.1 of [3] follows.

Finally, to see that also the series expansion for $\hat{h}_{ij}$ converges we note that, near i^0, Theorem 1.1 of [3] shows that the physical 3-metric g_{ij} exists. This means that $\lambda(h_G)_{ij}=\Omega^{-2}\hat{h}_{ij}=\frac{e^{-4\kappa}}{r^4}\hat{h}_{ij}$ exist as functions and in particular (after transvecting with x^j) that $\frac{e^{-4\kappa}}{r^4}x^i$ exist. This implies that the series expansions for κ, and thus for Ω, and therefore also for $\hat{h}_{ij}$, converges near i^0. □

3.8. The case $P^0=0$

To complete the proof, we must now relax the condition $m\neq 0$, and this will be possible due to the fact that the conformal factor $\hat{\Omega}$ is well behaved for $m=0$. Thus, suppose that we have a sequence of totally symmetric and trace-free tensors $P^0,P^0_{i_1},P^0_{i_1i_2},\ldots$ as in Theorem 1, and where $P^0=0$. In particular, it is assumed that $u(\mathbf{r})=\sum_{|\alpha|\geq 0}\frac{\mathbf{r}^\alpha}{\alpha!}P^0_\alpha$ converges in some polydisc $U:\{(x,y,z),|x|<d,|y|<d,|z[<d\}$. By replacing only the monopole, i.e., putting $P^0=m_0>0$, the corresponding sequence $m_0,P^0_{i_1},P^0_{i_1i_2},\ldots$ corresponds to the function $\tilde{u}(\mathbf{r})=m_0+\sum_{|\alpha|>0}\frac{\mathbf{r}^\alpha}{\alpha!}P^0_\alpha$, which also converges in U. Thus, by the arguments given so far there exists convergent, in a polydisc V say, power series

$$\hat{h}_{ij}=\sum_{|\alpha|\geq 0}(c_{ij})_\alpha\mathbf{r}^\alpha$$

for the metric components; furthermore we also know that there is a static spacetime having the multipole moments $m_0,P^0_{i_1},P^0_{i_1i_2},\ldots$. Now, from the recursion producing the metric, it is seen that each coefficient $(c_{ij})_\alpha$ is a polynomial in m_0, and hence the metric components h_{ij} can be regarded as a power series in the four variables (m,x,y,z):

$$\hat{h}_{ij}=\sum_{|\beta|\geq 0}(d_{ij})_\beta m^{\beta_1}x^{\beta_2}y^{\beta_3}z^{\beta_4},$$

where the multi-index $\beta=(\beta_1,\beta_2,\beta_3,\beta_4)$. Since this series converges for $m=m_0$, $(x,y,z)\in V$, it also converges for $|m|<m_0,(x,y,z)\in V$. We can now choose $m=0$ and still have convergence of $\hat{h}_{ij}$, κ and g_{ij}. In particular, with $m=0$ the multipole moments will be the initially desired sequence $P^0,P^0_{i_1},P^0_{i_1i_2},\ldots$ where $P^0=0$.

4. Acknowledgement

The author wants to thank Prof. Brian Edgar for helpful comments.

5. Discussion

In this paper, we have proved that the necessary conditions from [8] on the multipole moments of a static spacetime are also sufficient in order for a static vacuum spacetime having these moment to exist. In particular, we do not assume that the monopole is nonzero. This proves a long standing conjecture by Geroch, [2].

The conditions given are simple and natural. In essence, each allowed set of multipole moments is connected to a harmonic function on $\mathbf{R}^3$, defined in a neighbourhood of $\mathbf{0}$. The proof is constructive in the sense that an explicit metric having prescribed moments up to a given order can be calculated.

Considering future work, it is natural to see what can be carried over to the general stationary case. It may be conjectured that the natural generalisation of Theorem 1 is valid, with the extra condition that the monopole is real and nonzero. It could also be instructive to explicitly calculate the metric belonging to a static spacetime with arbitrary multipole moments up to a given order.

6. Appendix

Proof of Lemma 4. Again we make an ansatz

$$A = \begin{pmatrix} X(x,y,z) & a(x,y,z) & b(x,y,z) \\ a(x,y,z) & Y(x,y,z) & c(x,y,z) \\ b(x,y,z) & c(x,y,z) & Z(x,y,z) \end{pmatrix}$$

where a, b, c, X, Y and Y are analytic in the variables indicated. The condition $A_{ij}x^i = 0$ is

$$\begin{aligned} xX(x,y,z) + ya(x,y,z) + zb(x,y,z) &= 0 \\ xa(x,y,z) + yY(x,y,z) + zc(x,y,z) &= 0 \\ xb(x,y,z) + yc(x,y,z) + zZ(x,y,z) &= 0 \end{aligned} \tag{47}$$

Writing

$$X(x,y,z) = X_2(x,y) + z\ X_3(x,y,z), \quad a(x,y,z) = a_2(x,y) + z\ a_3(x,y,z),$$

(47) with $z = 0$ shows that $a_2(x,y) = x\ a_4(x,y)$, $X_2(x,y) = -y\ a_4(x,y)$ for some function a_4. Next it follows that $b(x,y,z) = -y\ a_3(x,y,z) - x\ X_3(x,y,z)$. After the decomposition $Y(x,y,z) = Y_2(x,y) + z\ Y_3(x,y,z)$, (47) with $z = 0$ shows that

$a_4(x,y) = y\ f_1(x,y)$, $Y_2(x,y) = -x^2 f_1(x,y)$ for some function f_1. Form this we find that $c(x,y,z) = -x\ a_3(x,y,z) - yY_3(x,y,z)$. Next we make the decomposition

$$\begin{aligned} a_3(x,y,z) &= a_6(x,y) + z\ f_5(x,y,z) \\ X_3(x,y,z) &= X_6(x,y) + z\ f_4(x,y,z) \\ Y_3(x,y,z) &= Y_6(x,y) + z\ f_6(x,y,z) \end{aligned}$$

form which (47), again with $z = 0$, shows that $X_6(x,y) = 2y\ f_2(x,y)$, $Y_6(x,y) = 2x\ f_3(x,y)$ for some function f_2 and f_3. Finally, it follows that $a_6(x,y) = -x\ f_2(x,y) - y\ f_3(x,y)$ and $Z(x,y,z) = 2xy\ f_5(x,y,z) + x^2 f_4(x,y,z) + y^2 f_6(x,y,z)$. By collecting terms, the lemma follows. $\square$

Proof of Lemma 11. Observe that the definition of u_i^a implies that $u_i^a(r^2) = 0$, $i = 1,2,3$. We now demonstrate that $t_{11} = u_1^a u_1^b T_{ab} \equiv 0 \pmod{r^2}$. From the definition of $\hat{R}_{ij}$,

$$u_1^i u_1^j R_{ij} = u_1^i u_1^j \frac{\partial}{\partial x^k}\Gamma^k{}_{ij} - u_1^i u_1^j \frac{\partial}{\partial x^i}\Gamma^k{}_{kj} + u_1^i u_1^j \Gamma^m{}_{ij}\Gamma^k{}_{mk} - u_1^i u_1^j \Gamma^m{}_{kj}\Gamma^k{}_{mi}$$

where $\Gamma^m{}_{ij} = \frac{1}{2}\hat{h}^{m\sigma}\{\frac{\partial \hat{h}_{i\sigma}}{\partial x^j} + \frac{\partial \hat{h}_{j\sigma}}{\partial x^i} - \frac{\partial \hat{h}_{ji}}{\partial x^\sigma}\}$ and $\Gamma^m{}_{im} = \frac{1}{2}\partial_i \ln|\hat{h}|$. Next, if B denotes any symmetric 3×3 matrix, we can write

$$\hat{h}_{ij} = \eta_{ij} + p_0 x_i x_j + r^2 B_{ij} \tag{48}$$

from which it follows that

$$\hat{h}^{ij} = \eta^{ij} - p_0 x^i x^j + r^2 B^{ij}.$$

Therefore, $\hat{h}^{m\sigma}\Gamma^k{}_{mk} \equiv (\eta^{m\sigma} - p_0 x^m x^\sigma)\frac{1}{2}\partial_m \ln|\hat{h}| \equiv \eta^{m\sigma}\frac{1}{2}\partial_m \ln|\hat{h}| \pmod{r^2}$, where we have used that $D(\ln|h|) \equiv 0 \pmod{r^2}$. Thus, $u_1^i u_1^j \Gamma^m{}_{ij}\Gamma^k{}_{mk} \equiv \frac{1}{4}u_1^i u_1^j\{\frac{\partial \hat{h}_{i\sigma}}{\partial x^j} + \frac{\partial \hat{h}_{j\sigma}}{\partial x^i} - \frac{\partial \hat{h}_{ji}}{\partial x^\sigma}\}\eta^{m\sigma}\partial_m \ln|\hat{h}| \pmod{r^2}$. Using that $\hat{h}_{ij}$ can be written as in (48) and that $u_1^i \partial_i x^j = u_1^j$ it follows that $u_1^i u_1^j \Gamma^m{}_{ij}\Gamma^k{}_{mk} \equiv 0 \pmod{r^2}$. A similar but longer calculation shows that also $u_1^i u_1^j \Gamma^m{}_{kj}\Gamma^k{}_{mi} \equiv 0 \pmod{r^2}$.

By expanding (11), using the above result, (20) and the relations (10), it amounts to proving that (using the form (18) for the metric)

$$\begin{aligned} &u_1^i u_1^j \Big[\partial_k \Gamma^k{}_{ij} - \partial_i \Gamma^k{}_{kj} + D(\gamma_{ij}) + 2\gamma_{ij} + 2\hat{D}_i\hat{D}_j\kappa \\ &+\eta_{ij}(2\Delta\kappa - D(f) - 2f - 8D(\chi) - 16\chi + D(\ln|\hat{h}|)/r^2)\ \Big] \equiv 0 \pmod{r^2} \end{aligned}$$

Next, by replacing $\kappa = r^2\chi$, it follows that $u_1^i u_1^j(2\hat{D}_i\hat{D}_j\kappa + \eta_{ij}(2\Delta\kappa - 8D(\chi) - 16\chi)) \equiv 0 \pmod{r^2}$ which leaves us with

$$u_1^i u_1^j \left[\partial_k \Gamma^k{}_{ij} - \partial_i \Gamma^k{}_{kj} + D(\gamma_{ij}) + 2\gamma_{ij} + \eta_{ij}(-D(f) - 2f + D(\ln|\hat{h}|)/r^2)\right] \tag{49}$$

With $\ln|\hat{h}| = r^2\sigma$, we get $u_1^i u_1^j \partial_i \Gamma^k{}_{kj} = u_1^i u_1^j \frac{1}{2}\partial_i\partial_j(r^2\sigma) \equiv \sigma u_1^i u_1^j \eta_{ij} \pmod{r^2}$. Next,

$$2u_1^i u_1^j \partial_k \Gamma^k{}_{ij} = u_1^i u_1^j \partial_k[\hat{h}^{k\sigma}\{\frac{\partial \hat{h}_{i\sigma}}{\partial x^j} + \frac{\partial \hat{h}_{j\sigma}}{\partial x^i} - \frac{\partial \hat{h}_{ji}}{\partial x^\sigma}\}]$$

and by using Leibniz's rule, one finds that

$$u_1^i u_1^j \partial_k(\hat{h}^{k\sigma})\{\frac{\partial \hat{h}_{i\sigma}}{\partial x^j} + \frac{\partial \hat{h}_{j\sigma}}{\partial x^i} - \frac{\partial \hat{h}_{ji}}{\partial x^\sigma}\} \equiv -u_1^i u_1^j \partial_k(\hat{h}^{k\sigma})\partial_\sigma(\hat{h}_{ij}) \pmod{r^2} \tag{50}$$

and

$$u_1^i u_1^j \hat{h}^{k\sigma} \partial_k\{\frac{\partial \hat{h}_{i\sigma}}{\partial x^j} + \frac{\partial \hat{h}_{j\sigma}}{\partial x^i} - \frac{\partial \hat{h}_{ji}}{\partial x^\sigma}\} \equiv u_1^i u_1^j [2\eta^{k\sigma}\partial_\sigma \partial_{(j}\hat{h}_{i)k} - \Delta_C \hat{h}_{ij}] \pmod{r^2} \tag{51}$$

Some further manipulations reveals that in (50), $u_1^i u_1^j \partial_k(\hat{h}^{k\sigma})\partial_\sigma(\hat{h}_{ij}) \equiv 0 \pmod{r^2}$ while from (51) we find $u_1^i u_1^j [2\eta^{k\sigma}\partial_\sigma \partial_{(j}\hat{h}_{i)k} - \Delta_C \hat{h}_{ij}] \equiv u_1^i u_1^j [8f\eta_{ij} + 6D(f)\eta_{ij} - 6\gamma_{ij} - 4D(\gamma_{ij})] \pmod{r^2}$ Inserted in (49), we get, modulo r^2,

$$u_1^i u_1^j \left[2f\eta_{ij} + 2D(f)\eta_{ij} - \gamma_{ij} - D(\gamma_{ij}) - \frac{1}{2}\partial_i \partial_j(\ln|\hat{h}|) + \eta_{ij}(D(\ln|\hat{h}|)/r^2)\right] \tag{52}$$

where also $u_1^i u_1^j \partial_i \partial_j(\ln|\hat{h}|) \equiv -z\partial_z(\ln|\hat{h}|) \pmod{r^2}$. To finish the proof, some linear algebra shows that $\ln|\hat{h}| = -2fr^2 + [\gamma_{ij}]r^2 \pmod{r^4}$. Finally an insertion in (52) gives, modulo r^2,

$$-u_1^i u_1^j(\gamma_{ij} + D(\gamma_{ij})) - z^2([\gamma_{ij}] + D([\gamma_{ij}])). \tag{53}$$

Now, for each degree in the series expansion of γ_{ij}, $D(\cdot)$ acts as multiplication operator, and therefore equation (53) $\equiv 0 \pmod{r^2}$ is equivalent to $u_1^i u_1^j \gamma_{ij} + z^2[\gamma_{ij}] \equiv 0 \pmod{r^2}$. However, γ_{ij} is composed as the sum of the matrices $B_1, B_2, \ldots, B_6$ in Lemma 9, and it is easily checked that $u_1^i u_1^j (B_k)_{ij} + z^2[(B_k)_{ij}] \equiv 0 \pmod{r^2}$ for $k = 1, 2, \ldots 6$, and therefore Lemma 11 follows. □

[1] Beig, R., Simon, W., *Proof of a Multipole Conjecture due to Geroch*, Commun. Math. Phys., **78**, 75 (1980).

[2] Geroch, R., *Multipole Moments. II. Curved Space*, J. Math. Phys., **11**, 2580 (1970).

[3] Friedrich, H., *Static Vacuum Solutions from Convergent Null Data Expansions at Space-Like Infinity*, Ann. Henri Poincaré **8**, 817-884 (2007)

[4] Hansen, R.O., *Multipole moments of stationary spacetimes*, J. Math. Phys., **15**, 46 (1974).

[5] Herberthson, M., *The gravitational dipole and explicit multipole moments of static axisymmetric spacetimes*, Class. Quantum Grav. **21**, 5121 (2004).

[6] Bäckdahl, T., Herberthson, M., *Static axisymmetric spacetimes with prescribed multipole moments*, Class. Quantum Grav. **22**, 1607 (2005).

[7] Bäckdahl, T., Herberthson, M., *Explicit multipole moments of stationary axisymmetric spacetimes*, Class. Quantum Grav. **22**, 3585 (2005).

[8] Bäckdahl, T., Herberthson, M., *Calculation of, and bounds for, the multipole moments of stationary spacetimes*, Class. Quantum Grav. **23**, 5997 (2006).

[9] Quevedo, H., *Multipole Moments in General Relativity - Static and Stationary Vacuum Solutions*, Fortschritte der physik, **38**, 733 (1990).

[10] Simon, W., Beig, R., *The multipole structure of stationary spacetimes*, J. Math. Phys. **24**, 1163 (1983).

[11] Thomas, T.Y., *Concepts from Tensor Analysis and Differential Geometry*, Academic Press, 1961.

[12] Wald, R.M., *General Relativity*, The University of Chicago Press, Chicago 1984.